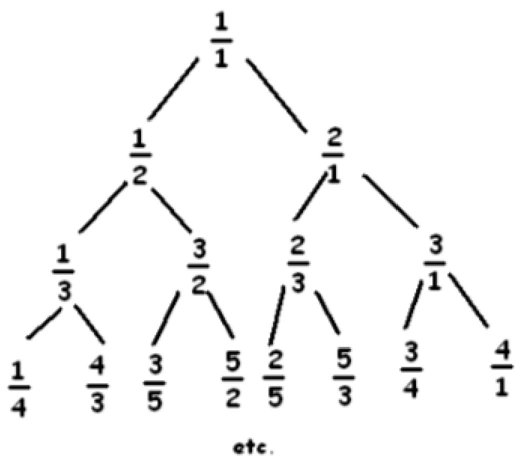
Fraction Trees[1]

1 Fraction Tree

Consider the following fraction tree



1. Each fraction has two "children:" a left child and a right child . What is the rule for generating each of these children?
2. Continue drawing the fraction tree for another two rows.
3. Explain why the fraction $\frac{13}{20}$ will eventually appear in the tree. (It might be easier to first figure out what should be the parent of $\frac{13}{20}$, and the grandparent, etc.)
4. Might the fraction $\frac{13}{20}$ appear twice in the tree?
5. Will the fraction $\frac{457}{777}$ eventually appear in the tree? Will it appear twice?
6. Are all the fractions that appear in the tree in reduced form? (For example, I see the fraction $\frac{2}{3}$, but I don't see its equivalents $\frac{4}{6}$ or $\frac{12}{18}$.)
7. Prove that this tree produces all possible positive fractions, written in reduced form, with no fraction ever appearing more than once.

[1]Copied from James Tanton's THINKING MATHEMATICS www.jamestanton.com

2 Euclidean Algorithm

8. Find the greatest common divisor of 84 and 120.
9. True or False: if a divides b and c, then a divides c and $b - c$.
10. True or False: If a divides c and $b - c$, then a divides b and c.
11. True or False: The common factors of 120 and 84 are the same as the common factors of 84 and $120 - 84 = 36$.
12. True or False: The common factors of 84 and 36 are the same as the common factors of 36 and $84 - 36 = 48$.
13. Explain how the above statements and similar additional statements can help you find the greatest common factor of 84 and 120.
14. What is the greatest common divisor of 949 and 2701?
15. What is the common divisor of 532 and 747?
16. How does the Euclidean algorithm for finding greatest common divisors help you locate $\frac{532}{747}$ in the fraction tree?

3 Fraction List

17. If we read the fractions left to right across the rows of the tree we obtain $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}. \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \cdots /$

    Look at the numerators that appear in the fraction tree, reading across the tree by its rows. We obtain the sequence:

    S = 1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 5...

    What sequence do you get if you write down the denominators and how does it compare to the sequence of numerators?

18. Pluck out every second entry from sequence S of numerators, the ones in even positions. How does this sequence compare to S?

19. What do you notice about the odd numbers in sequence S and their neighbors?

20. Which entries in S are even and which are odd?

21. Write down the numbers in S that appear between the 1's. What do you notice about them?

22. Can you prove that these observations always hold?

4 Insert Sum Here

Consider the following table of numbers.

Row 1: 1 1
Row 2: 1 2 1
Row 3: 1 3 2 3 1
Row 4: 1 4 3 5 2 5 3 4 1
Row 5: 1 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 1

23. Describe the pattern: if we have one row, how do we create the next row?
24. How many numbers are in row 1? How many numbers in row 2? How many numbers in row 3? What's the pattern? Why will it continue?
25. A lot of people working with this problem don't count the 1 at the end of each row. Would that make some of your answers here simpler?
26. How many 1s are in row 1? How many 2s are in row 2? How many 3s are in row 3? How many 4s are in row 4? How many 5s are in row 5? Can you find a pattern here?
27. How is this "insert sum here" sequence related to the list of numerators from the fraction tree?

5 Hyperbinary representations

28. I have two rocks each weighing 1 pound, two rocks each weighing 2 pounds, two each weighing 4 pounds, two each weighing 8 pounds, and so on. In how many ways can I make ten pounds?

29. When you write a number as a sum of *distinct* powers of 2, that is called a *binary representation*. For example, $13 = 8 + 4 + 1$ is a binary representation of 13. This can also be communicated by writing $13 = 1101_2$. Here, the first 1 mean one 8, the second 1 means one 4, the 0 means zero 2's, and the 1 means one 1.

    Write a binary representation of 23.

30. When you write a number as a sum of powers of 2, where you can use each power of 2 up to two times, this is called a *hyperbinary* representation of the number. For example, 6 can be written as $4 + 2$ but it can also be written as $4 + 1 + 1$. Are there any other hyperbinary representations of 6?

31. What are all the hyperbinary representations of 10?

32. Write down the number of hyperbinary representations of 1, 2, 3, 4, 5, etc. in a list. Does this list remind you of any of the previous lists that we have considered?

6 Continued Fractions

33. For a given number N, write N in binary and count up the length of the strings of 1's, and 0's going from right to left, starting with strings of 1's.

    For example, for 13, since $13 = 1101_2$, we would count 1, 1, 2 for the strings of 1's, 0's and 1's. For 100, since $100 = 1100100_2$, we would count 0, 2, 1, 2, 2.

    Next, create a continued fraction from this list of counts.

    For 13, that would be $f_{13} = 1 + \cfrac{1}{1 + \cfrac{1}{2}}$ which simplifies to $\frac{5}{3}$

    For 100, that would be

    $$f_{100} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{7}{19}$$

34. Compute f_1, f_2, f_3, etc. What do these fractions have to do with the original fraction tree or fraction list?

35. Can you predict the 100th term in the fraction list without writing out lots of lines of the tree? What about the 1000th?

36. What is your method of finding the nth term?

37. Can you prove that this method works? Hint: think of how the continued fraction representations of the children compare to that of the parent. Is there a similar relationship between the binary representations of children and their parent in this tree?