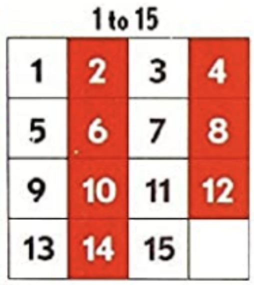
Permutations and Slide Puzzles

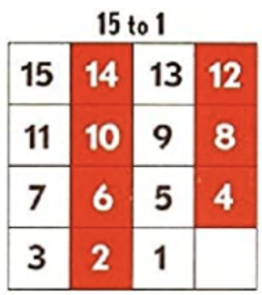
Play

1. The goal of the 15-puzzle is to slide the squares around to get to various configurations.

   (a) Can you get to the 1 to 15 configuration?

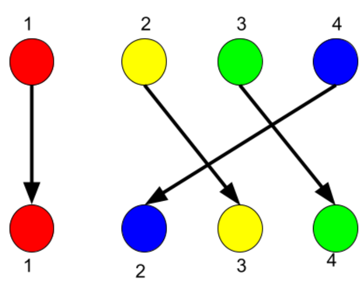
   (b) Can you get to the 15 to 1 configuration?

   Like the Rubiks Cube, a slide puzzle can be described in terms of permutations.

Permutations

A permutation is a rearrangement of objects. For example, suppose you have 4 balls, numbered 1, 2, 3, 4.



This permutation can be described with numbers like this:

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$

Or like any of these:

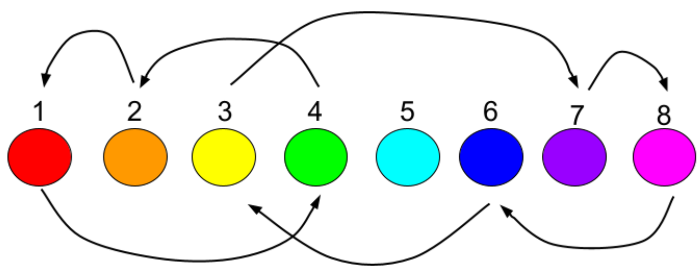
$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ or $\begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

A more efficient way to describe the permutation is using *cycle notation*:

(1)(234)

The set of parentheses around the 1 means that 1 goes to itself. The notation (234) means that 2 gets sent to 3, 3 gets sent to 4, and 4 gets sent to 2 (we wrap around to the beginning). A permutation like (234) is called a length 3 cycle or a 3-cycle.

2. Describe this permutation



   (a) in two-row notation

   (b) in cycle notation

3. Convert

   (a) $\alpha = (1\ 7\ 2\ 5)(3\ 4\ 6\ 9)(8)$ to two-row notation.

   (b) $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 5 & 6 & 8 & 3 & 7 \end{pmatrix}$ to cycle notation.

4. Which of the following represent the same permutation?

   (a) (1)(234) (b) (1)(342) (c) (234)(1) (d) (432)(1) (e) (43)(12) (f) (21)(34)

   It is common to omit cycles of length one from our notation. So if we know there are 5 things being permuted, (234) means the same thing as (1)(234)(5).

   It is also common to order the numbers within each cycle in from smallest to largest and to order the cycles by their first entry from smallest to largest. So (4523)(761) would be written as (176)(2345).

5. Often, permutations are written with *disjoint cycles*. That is, the numbers in each cycle are different. Which of these permutations have disjoint cycles?

(a) (24)(23)
(b) (132)(23)
(c) (24)(13)

6. Write the following permutations with disjoint cycles.

   (a) (24)(23)
   (b) (132)(23)

7. Can any permutation be written with disjoint cycles?

8. Sometimes, it is handy to write a permutation as a bunch of 2-cycles (swaps), which are not necessarily disjoint. Write the following permutations as products of 2-cycles. How many 2-cycles did you use?

   (a) (12345)
   (b) (1578)(243)(6)
   (c) (243)(12)(142)
   (d) (123)(342)(134)
   (e) (142)(23)(341).

9. Is it possible to write the permutation (243)(12)(142) with one 2-cycle? two 2-cycles? three 2-cycles? four 2-cycles? five 2-cycles? seventeen 2-cycles?

10. What numbers of 2-cycles can you use to write the permutation (123)(342)(134)?

11. It turns out that a permutation that can be written as an even number of 2-cycles, say with six 2-cycles, can only be written with even numbers of 2-cycles. It could be written with eight or ten 2-cycles, and might possibly be written as four or two or even zero 2-cycles, but it can never be written as three or five or one or seven 2-cycles.

    Similarly, a permutation that can be written as an odd number of 2-cycles can only be written with an odd number of 2-cycles, never an even number.

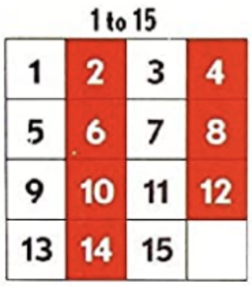
    A permutation is called *even* if it can be written with an even number of 2-cycles, and *odd* if it can be written with an odd number of 2-cycles.

12. For each permutation below, decide if the permutation is even or odd.

    (a) (234)
    (b) (56)
    (c) (17453)
    (d) (345)(256)
    (e) (17453)(26)(8)
    (f) (1652)(3478)(96)
    (g) (36748125)

Permutations on the 15-Puzzle

Any position of the 15-puzzle can be thought of as a permutation from the "starting position" shown below, where the blank square is thought of as square 16.



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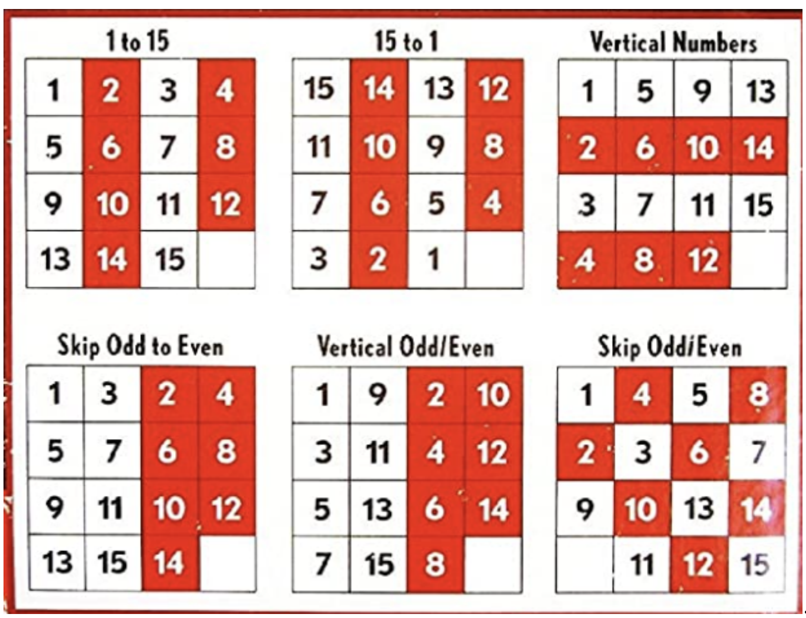
13. Write each of the following configurations as a permutation of the starting configuration, using only 2-cycles. Which ones are even permutations and which are odd permutations?



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14. Even and odd permutations is the key to which positions are solvable and which are not. What is your conjecture about which positions are solvable?

15. In the game, we can't do every kind of 2-cycle swap, because we aren't allowed to pick up the tiles from the board and move them through the air. The only kind of swap we can do is swapping the blank square with an adjacent square.

    Suppose we do a sequence of moves, and the blank square starts and ends in the bottom right corner.

    (a) If the sequence of moves includes exactly 5 swaps that move the blank square up one, how many swaps move the blank square down one?

(b) If the sequence of moves includes exactly 8 swaps that move the blank square left by one, how many swaps move the blank square right by one?

16. If a sequence of moves begins and ends with the blank square in the bottom right corner, then what do you know about the total number of swaps?

17. If a sequence of moves begins and ends with the blank square in the bottom right corner, what do you know about the even or odd status of the permutation?

18. Which of the configurations shown above are impossible to solve? Explain in terms of even and odd.

Algorithm for Solving the Possible Configurations

In fact, it is not so hard to find an algorithm to solve any of the other configurations. To do this, it is helpful to find a few "macros", or sequences of moves that make a change to only a few tiles.

19. Experiment with some sequences of moves that involve only six tiles in a 2 x 3 rectangle that includes the blank tile. Try to find some useful macros.

20. Is it possible to find a macro that swaps exactly two tiles?

21. Is it possible to find a macro that rotates exactly three tiles?

22. Use the macros you find to solve the 15 puzzle for the solvable positions.