Knowledge about Knowledge

1 Hat Puzzles

The kids in the room have red or white hats. They can see each other's but not their own. Every whole hour on the hour, whoever knows their own color stands up and says "Got it!" (if a kid figures out their own hat color as a result of some other kid standing up, they must keep a poker face and wait for the next round to announce it). All of these rules are common knowledge.

1. Two kids, one red hat, one white hat. Does anything ever happen? (no). A person walks in the room at 12:30 and says "Oh, nice to see one-or-more red hats". What happens?

2. Two kids, both have red hats. A person walks in and makes the same statement. What happens? How can that be, given that both kids knew about a red hat long before the statement was made?

3. Three kids, all have red hats. A person walks in and makes the same statement. What happens?

4. Ten kids, 3 have red hats, 7 have white hats. A person walks in and makes the same statement. What happens?

2 More Hat Puzzles

These are copied from Wikipedia's hat puzzles article

`https://en.wikipedia.org/wiki/Hat_puzzle`

5. Three Hats in a Circle: Three players are told that each of them will receive either a red hat or a white hat. They are to raise their hands if they see a red hat on another player as they stand in a circle facing each other. The first to guess the color of his or her hat correctly wins.

   All three players raise their hands. After the players have seen each other for a few minutes without guessing, one player announces "red", and wins. How did the winner do it, and what is the color of everyone's hats?

6. Four Hats in a Line: Four prisoners are arrested for a crime, but the jail is full and the jailer has nowhere to put them. He eventually comes up with the solution of giving them a puzzle so if they succeed they can go free but if they fail they are executed.

   The jailer seats three of the men into a line. B faces the wall, C faces B, and D faces C and B. The fourth man, A, is put behind a screen (or in a separate room). The jailer gives all four men party hats. He explains that there are two red hats and two white hats, that each prisoner is wearing one of the hats, and that each of the prisoners see only the hats in front of him but neither on himself nor behind him. The fourth man behind the screen can't see or be seen by any other prisoner. No communication among the prisoners is allowed.

If any prisoner can figure out what color hat he has on his own head with 100% certainty (without guessing) and tell the jailer, all four prisoners go free. If any prisoner suggests an incorrect answer, all four prisoners are executed. How can the prisoners escape?

7. Four Hats with Three in a Circle: In this version the prisoners know that there are. 3 red hats and 1 white hat, and the 3 prisoners can see each other i.e. D sees B and C, B sees D and C, and C sees D and B. (A again cannot be seen and is only there to wear the last hat.)

8. Five Hats: In the next variant, only three prisoners and five hats (supposedly two red and three white) are involved. The three prisoners are ordered to stand in a straight line facing the front, with A in front and C at the back. They are told that there will be two red hats and three white hats. One hat is then put on each prisoner's head; each prisoner can only see the hats of the people in front of him and not on his own. The first prisoner that is able to announce the color of his hat correctly will be released. No communication between the prisoners is allowed.

9. Three Hats at Random: In this variant there are 3 prisoners and 3 hats. Each prisoner is assigned a random hat, either red or white. In all, there are three red hats and two white. Each person can see the hats of two others, but not their own. On a cue, they each have to guess their own hat color or pass. They win release if at least one person guessed correctly and none guessed incorrectly (passing is neither correct nor incorrect).

   This puzzle doesn't have a 100% winning strategy, so the question is: What is the best strategy? Which strategy has the highest probability of winning?

10. An even odder 10 hat puzzle: In this variant there are 10 prisoners and 10 hats. Each prisoner is assigned a random hat, either red or white, but the number of each color hat is not known to the prisoners. The prisoners will be lined up single file where each can see the hats in front of him but not behind. Starting with the prisoner in the back of the line and moving forward, they must each, in turn, say only one word which must be "red" or "white". If the word matches their hat color they are released, if not, they are killed on the spot. A sympathetic guard warns them of this test one hour beforehand and tells them that they can formulate a plan where by following the stated rules, 9 of the 10 prisoners will definitely survive, and 1 has a 50/50 chance of survival. What is the plan to achieve the goal?

11. Ten Hats without Hearing: As before, there are 10 prisoners and 10 hats. Each prisoner is assigned a random hat, either red or blue, but the number of each color hat is not known to the prisoners. The prisoners are distributed in the room such that they can see the hats of the others but not their own. Now, they must each, simultaneously, say only one word which must be "red" or "blue". If the word matches their hat color they are released, and if enough prisoners resume their liberty they can rescue the others. A sympathetic guard warns them of this test one hour beforehand. If they can formulate a plan following the stated rules, 5 of the 10 prisoners will definitely be released and be able to rescue the others. What is the plan to achieve the goal?

3 The Forehead Consecutive Number Game

Annie and Zoe have consecutive natural numbers written on their foreheads. This is common knowledge (they both know this, and they both know they know this, etc.) We ask them in turn if they know their own number: "Annie, do you know?" - "Zoe, do you know?" - "Annie, do you know now?"- etc. They only answer "Yes, I do know" or "No, I don't", and we keep asking until someone figures out their own number (it turns out that the other person then does, too, right on the next question).

12. If Annie has 2 and Zoe has 1, what happens?
13. If Annie has 1 and Zoe has 2, what happens?
14. If Annie has 3 and Zoe has 2, what happens?
15. If Annie has 2 and Zoe has 3, what happens?
16. If Annie has 4 and Zoe has 3, what happens?
17. If Annie has 3 and Zoe has 4, what happens?
18. If Annie has 100 and Zoe has 101, what happens?

4 The Forehead Sum Game

Alice, Bob and Charlie have positive integers x, y, z on their foreheads with $x + y = z$. This is common knowledge. We ask them in turn if they know their own number.

19. Alice=1, Bob=1, Charlie=2: what happens? How about 10, 10, 20? Generally, what happens if two numbers are the same?
20. Alice=1, Bob=2, Charlie=3: what happens? How about 10, 20, 30? Generally, what happens if one number is double another?
21. Alice: "No". Bob: "No". Charlie: "No". Alice: "No". Bob: "No". Charlie: "Yes, it's 12". What are the numbers?
    - We're trying to determine the set of numbers on Alice's and Bob's foreheads. We don't expect to determine which of them has which number.
    - This is a somewhat harder problem. Think about what you've learned, and what ratio between numbers would yield two rounds of "No" for Bob.

5 Sum and Product Puzzles

22. Someone picks two integers, X and Y, each from the interval 2 to 99, and tells Mr. Sam the sum of the two integers, and Mr. Paul the product of the two. Sam and Paul do not know the values given to the other. Paul tells Sam, "I do not know the two integers." Sam tells Paul, "I knew you wouldn't. Neither do I." Paul replies, "Oh, now I know the integers." Sam replies back, "Now I know too."

    Given that both of them are telling the truth, what could the two integers be?

23. Here's another variant, even harder: P and S are given the product and sum of two non-zero digits (1 to 9). It common knowledge that P will know the product and S will know the sum.

    (a) P says "I don't know the numbers."
    S says "I don't know the numbers."

    (b) P says "I don't know the numbers."
    S says "I don't know the numbers."

    (c) P says "I don't know the numbers."
    S says "I don't know the numbers."

    (d) P says "I don't know the numbers."
    S says "I don't know the numbers."

    (e) P says "I know the numbers."

This week's problems are from Alon Amit, compiled from various sources, and from Wikipedia's Hat Puzzle article.