Permutations and the Rubiks Cube [1]

Getting started

Definitions and notation.

- The little cubes that make up the Rubiks cubes are the *cubies*. How many are there?
- There are three types of cubies:
  1. *center cubies*, that lie in the center of each face,
  2. *edge cubies* that lie in the middle of edges, and
  3. *corner cubies*

  How many of each type of cubie are there?
- The visible faces of the cubies are called *facelets*. How many facelets are there?
- We will call the faces of the cube: front, back, left, right, up, down.
- We will use the first letters of those words F, B, L, R, U, D to describe a *quarter turn clockwise twist* about the front, back, left, right, up, and down faces, respectively.
- We will use F', B', L', R', U', D' to describe a quarter turn counterclockwise twist.
- To refer to individual cubies, we will extend the same notation, using lower case letters. So u is the center cubie on the up face, ur is the edge cube along the edge between the up and right faces, and luf is a corner cubie.
- We will call a sequence of moves, done in order a *macro*. For example, $FRBR'$is a macro.
- You can find a Rubiks Cube simulator at https://rubikscu.be/

Encoding the cube mathematically

1. Look closely at an F twist. How many of the 54 facelets are rearranged by this twist? How about a U twist?
2. If you perform an F twist, you can undo it with an F'.
   (a) If we perform FR, what move(s) will undo it?
   (b) How do we undo $FRF'R'$?
   (c) How do we undo $F'F'RU'DL'U$?

[1]from A Decade of the Berkeley Math Circle - Section 3 by Tom Davis

3. We will call the do nothing move 1. What is $F1$? $B1$? What is FF'? Why is 1 a good symbol for the do nothing move?

4. Are twists *communtative*? That is, if we do, for example, FR, does that give us the same result as RF?

5. Are there any pairs of moves that commute? Pairs of macros?

6. The exponent notation can be used to write repeated moves. For example F^3 means FFF.

   (a) Why is $F^9 = F$?

   (b) Rewrite F^6 in simpler form.

   (c) What is B^8?

Order of macros

7. The order of a macro is the smallest number of times you have to do the macro before you get back to the original cube state. What is the order of R? What is the order of $FFRR$?

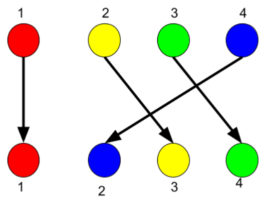
8. Does every macro have a finite order? Or are there some that will keep jumbling up the cube and never get it back to the original state, no matter how many times you apply the macro?

9. What happens if you apply the macro $FFRR$ only half as many times as its order? How could this be useful?

10. One way to find useful macros is to try a short set of moves and find its order k. If the order k is divisible by a small number, say 2, or 3, or maybe 5, then divide k by that number and apply the macro a number of times equal to this quotient. Try to find other useful macros this way.

Permutations

A permutation is a rearrangement of objects. For example, suppose you have 4 balls, numbered 1, 2, 3, 4.



This permutation can be described with numbers like this:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

Or like any of these:

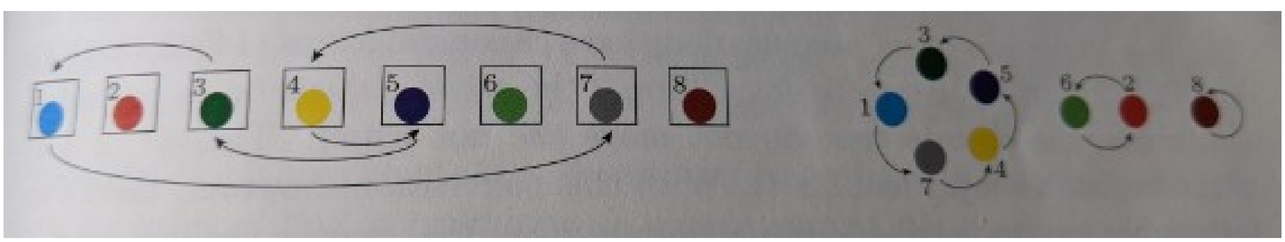
$$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

A more efficient way to describe the permutation is using *cycle notation*:

(1)(234)

The set of parentheses around the 1 means that 1 goes to itself. The notation (234) means that 2 gets sent to 3, 3 gets sent to 4, and 4 gets sent to 2 (we wrap around to the beginning). A permutation like (234) is called a length 3 cycle or a 3-cycle.

11. Describe this permutation



(a) in two-row notation

(b) in cycle notation

What cycles does it have and what are their lengths?

12. Convert

    (a) $\alpha = (398)(14562)(7)$ to two-row notation.

    (b) $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 6 & 7 & 8 & 4 & 5 \end{pmatrix}$ to cycle notation.

13. Which of the following represent the same permutation?

    (a) (1)(234) (b) (1)(342) (c) (234)(1) (d) (432)(1) (e) (43)(12) (f) (21)(34)

    It is common to omit cycles of length one from our notation. So if we know there are 5 things being permuted, (234) means the same thing as (1)(234)(5).

    It is also common to order the numbers within each cycle in from smallest to largest and to order the cycles by their first entry from smallest to largest. So (4523)(761) would be written as (176)(2345).

14. What is the order of the following permutations?

    (a) (234)
    (b) (56)
    (c) (17453)
    (d) (347)(256)
    (e) (17453)(26)(8)
    (f) (1652)(3478)(910)
    (g) (36748125)

15. Note that for the orders to work like this, the cycles have to be disjoint, rather than overlapping like (23)(31). Rewrite each of the following permutations with disjoint cycles and find its order. Work from left to right.

    (a) (23)(31)
    (b) (234)(17453)

16. (a) What is the order of the permutation $P = (12)(34567)$?

    (b) Which numbers doe the permutation P^5 move in the end?

    (c) Write P^5 in cycle notation.

Macros for the Rubiks cube

17. Using the individual cubies' names (e.g. u, ur, flu), write the cycle structure of the macro $FFRR$. Explain why $(FFRR)^3$ is a product of two disjoint 2-cycles.

18. Sometimes a macro will leave cubies in the same location, but rotated (a corner cubie) or flipped (an edge cubie).

    (a) What happens when you perform $LD'L'F'D'FUF'DFLDL'U'$?

    (b) What happens when you perform $L'U'LU'L'U'U'LRUR'URUUR'$?

    Even if the cubies move to different positions, they may also rotate around, and the current notation does not express this.

    To specify how cubies get twisted or flipped in place, or when moved to different positions, we could list where every facelet of every cubie moves. But there are 54 facelets, so this could be a long list. Easier is to indicate flips and rotations like this:

    - (uf) means that the uf edge cubie stays in place, but is flipped
    - (urf) means that the urf corner cubie is twisted in place where the up facelet moes to the right facelet, the right to the front, and the front to the up facelet.

19. What is the order and the standard cycle structure of $FUL'L'R$?

20. What is the cycle structure of $(FUL'L'R)^9$?

21. What is the cycle structure of $(FUL'L'R)^{18}$?

22. Investigate the properties of the move $FRF'R'$. Apply multiples of it and see if you can generate useful macros from it.

23. Find other macros that would be useful.