The Fourth Dimension

1 Coordinate Systems

1. What dimension is the space consisting of all points on the surface of the chalkboard? Why?
2. What dimension is the inside of a circle? The boundary of a circle? Why?
3. What dimension is the space of all colored pixels on a standard RGB computer screen?

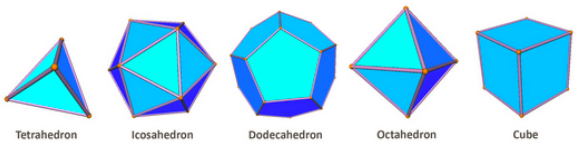
2 Distances

4. What is the distance between the points $(2019, 2020)$ and $(2023, 2025)$ on the plane? What is the distance between $(-1, 2)$ and $(3, 5)$? In general, how do we compute distance in the plane?
5. What is the distance from the origin $(0, 0, 0)$ to the point $(1, 2, 3)$ in three dimensional space? What is the distance between the points $(2, 5, 11)$ and $(3, 7, 13)$?
6. What is the distance between the points $(1, -1, 0, -1)$ and $(0, 1, 4, 9)$ in four dimensional space? Find two points in four dimensional space with integer coordinates that are exactly seven units apart.

3 Polyhedra

7. A polygon is 2 dimensional figure with straight sides. Here are some regular ones:

A polyhedron is a 3-dimensional figure with flat faces that are polygons. Here are some regular ones:



Fill in the blank: A 4-dimensional polytope is a 4-dimensional figure with ....

Are there any 1-dimensional analogs of polygons? What would be the regular ones?

8. Which 3-d polyhedron is most like a 2-d square? Suppose you were trying to describe this polyhedron to a 2-dimensional creature who couldn't imagine 3-dimensions. How could you describe it?

9. What is the 4-d analog of a cube? How many vertices (V), edges (E), faces (F), and hyperfaces (H) does it have? It may be helpful to think about how you could generate a cube beginning with a square, and then reason by analogy.

10. How would you draw a picture of a cube to show to a 2-dimensional creature? Can you draw a picture of a hypercube?

11. Which 3-d polyhedron is most like a 2-d triangle? How would you describe it to a 2-dimensional creature?

12. What is the 4-d analog of a tetrahedron? How many vertices (V), edges (E), faces (F), and hyperfaces (H) does it have? It may be helpful to think about generating a tetrahedron from a triangle first, and then reason by analogy.

13. For polyhedra, Euler's formula says that $V - E - F = 2$. Is there an analogous Euler's formula for 3-d polytopes? What should it say?

14. Can you generalize the square and the triangle to 5 dimensions?

15. Build a model of a 4-dimensional cube using Zome tools. You will actually be building the projection, or "shadow", of its edges in 3-dimensions.

16. How could you unknot a trefoil knot in 4-dimensions?

17. One morning, I woke up and got dressed. When I went to put on my shoes, both shoes were left shoes. I immediately suspected that a 4-dimensional creature had gotten in my house and played a trick on me. How did he get in my house and how did he turn my right shoe into a left one?

18. Is it possible for an ant to begin at one vertex of a cube, crawl along all the edges, and then return to its starting point without retracing any portion of its path? Is this possible on a hypercube? (Make sure that you have an accurate hypercube sketch first.)

19. A ball of radius r consists of the set of all points whose distance from a given point (the center of the ball) is r or less. Describe how a ball looks in one, two, or three dimensions. Then state a formula for the length, area, or volume of such a ball. Based on these expressions, what sort of formula would you expect for the volume of a four-dimensional ball of radius r?

Problems from *Circle in a Box* by Sam Vandervelde

4 4-Dimensional Platonic Solids

A *polytope* is a polyhedron in any dimension, not necessarily dimension 3.

- A 4-dimensional polytope has vertices, edges, faces and 3-dimensional "hyperfaces" (V, E, F, H).
- A 5-dimensional polytope has vertices, edges, faces, hyperfaces, and 4-dimensional "spaces" (V, E, F, H, S).

A Platonic solid in 3-dimensions is a polyhedron whose faces are all the same regular polygon and whose vertices all have the same number of faces around them.

20. How would you define Platonic solids in 4-dimensions? Can you give some examples?

21. These are some vital statistics for the 4-d Platonic solids.

V	E	F	H
5	10	10	5
16	32	24	8
8	24	32	16
24	96	96	24
600	1200	720	120
120	720	1200	600

They are called the 4-simplex, the hypercube, the 4-orthoplex, the 24-cell (or octaplex), the 120-cell, and the 600-cell.

What patterns do you notice?

22. Build models of the 4-dimensional Platonic solids. You will actually be building their *projections*, or shadows, in 3-dimensions.

23. Find V, E, F, H for a pyramid over a 3-dimensional tetrahedron, cube, icosahedron, and dodecahedron.

24. Find V, E, F, H for a bipyramid over an octahedron.

25. What about V, E, F, H for a prism over a cube? Think about why this is the same thing as a square"times" a square.

26. What are V, E, V, H for a pentagon "times" a pentagon. Can you find formulas for V, E, F, H for the product of an m-gon and an n-gon?

And Beyond

27. What Platonic solids can you describe in 5 dimensions?

28. Calculuate V, E, F, H, S for a 5-dimensional polytope, where S is the number of 4-dimensional spaces, and use these numbers to find the Euler characteristic $V - E + F - H + S$.

29. Which polytopes generalize easily to every dimension? What is Euler's formula in dimension n?

Thanks to Matt Kahle for most of these problems.