The Game of Set[1]

Getting started

1. Each card in the game of SET has symbols characterized by four different attributes:
   - Number: 1, 2, or 3 symbols
   - Color: red, green, or purple
   - Shading: empty, striped, or solid
   - Shape: ovals, squiggles, or diamonds

2. How many cards are in the deck? See if you can answer without counting.

3. Three cards make a "SET" if they are either all the same or all different in each of the four attributes. Which of the examples below make SETs and which do not? Explain.

4. To play the game,
   (a) Find a partner.
   (b) Lay out 12 cards in a 3 x 4 grid.
   (c) If you find a SET, say "SET" and pick it up. Then lay out another three cards to fill in the grid.
   (d) If someone calls "SET" but it isn't one, then they have to stay silent while the other player takes their time to find a true set.
   (e) If neither player can find a SET, then lay out another three cards and keep trying.
   (f) Whoever has the most SETs at the end wins.

   Play a game with a classmate.

Counting SETs

5. How many SETs are there that contain the card with 3 solid green ovals?

6. How many SETs are there in the whole deck?

7. How many different kinds of SETs are there? (Different kinds here means they have different numbers of attributes the same vs. different. For example, a SET which has three attributes the same and only one different is different from a SET that has all four attributes different.)

8. How many of each kind of SET are there in the deck?

[1] from The Joy of Set by McMahon, Gordon, Gordon, & Gordon

SETS and Geometry

9. Try this trick:

   (a) Start with three random cards that do NOT form a SET.

   (b) For each pair of those cards, find the third card that makes a SET with that pair.

   (c) Now do the same thing with the three cards you just found.

   (d) Now do the same thing with the three cards you just found. What happens?

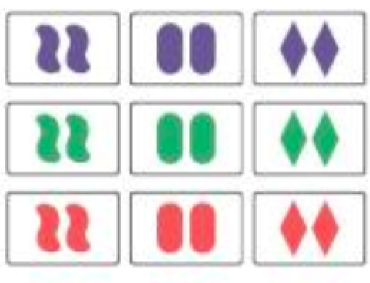
   (e) Lay out the cards nicely so that you can see all the SETs.

   (f) We will call this formation a plane. Notice that no matter which two cards you pick, the third card that makes a SET is in the plane!

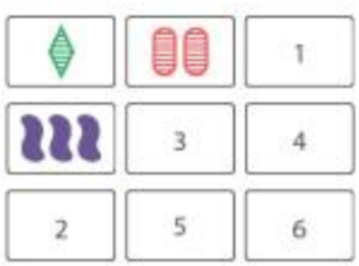
10. How many cards are in the plane you just made?

11. How many SETs are there in total in the cards you just laid out?

12. How many SETs are there in the plane drawn below? Draw a line through each triple of cards that makes a SET.



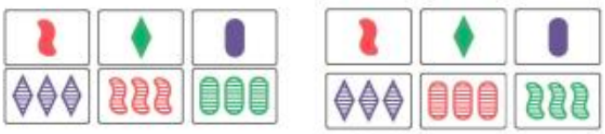
13. Complete the plane.



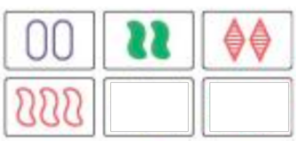
14. True or False and explain:

   - Given any two cards, there is a unique third card that complete a SET with them.
   - True or False and explain: Given any three cards that don't form a SET, there is a unique plane (up to reordering the cards) containing them.

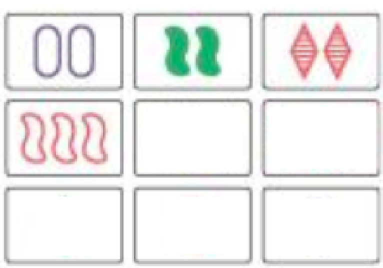
15. Compare the previous two statements to the following two statements:
    - Given any two points, there is a unique line containing them.
    - Given any three non-collinear points, there is a unique plane containing them.
16. What is a "line" in SET? What is a "point"?
17. Is there an analogy to "parallel lines" in SET?
18. Which of these is an example of two parallel lines?



19. In the figure below, find the line that is parallel to the top line and starts with the card shown in the second row.



20. Complete the plane.



Affine Planes

Here are the axioms of Finite Affine Plane Geometry:

Axiom 1 There are at least three non-collinear points.

Axiom 2 Every line contains at least two points.

Axiom 3 Two points determine a unique line.

Axiom 4 For any line ℓ and any point P not on ℓ, there is exactly one line containing P and not containing any point on ℓ. This line is said to be parallel to ℓ.

21. Verify that these axioms hold for a plane in SET. Therefore, a plane in SET is an affine plane.

22. Use the axioms of affine plane geometry to prove that for three lines r, s, and t, if r is parallel to s and s is parallel to t, then r is parallel to t.

    Hint: If r were not parallel to t, they would have a point of intersection. Now use Axiom 4 with that point and s to get a contradiction.

23. Use the axioms of affine plane geometry to prove that all lines have the same number of points.

    Hint: Take two distinct lines r and s and work through the following steps.

    (a) There is a point A on r that is not on s. Why / by which axiom(s)?

    (b) There is a point B on s that is not on r. Why?

    (c) Draw a line ℓ through A and B. How do you know there is a line / by which axiom(s)?

    (d) Now for another point on r besides A, draw a line through the point parallel to ℓ. Why can you do this / by which axiom(s)?

    (e) This line will intersect s in one point (not zero, and not more than one). Why / by which axioms?

    (f) Match up these two points that are on this parallel line: the one on r and the one on s.

    (g) For each point on r and each point on s, continue to match up points this way.

    (h) Why does this show that r and s have the same number of points?

24. Prove that in an affine plane geometry with n points on a line, there are $n+1$ lines through a point.

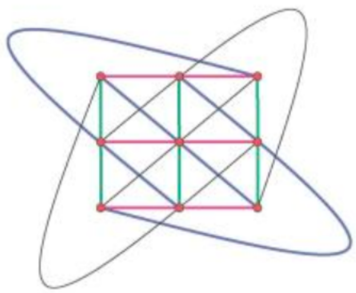
    Hint: for a point P, find a line ℓ that does not contain P (how do you know one exists?). Every line through P either intersects ℓ or doesn't. Count them up.

25. In an affine plane geometry with n points on a line, how many points total are there? Prove it!

    Hint: first prove that there are at least that many points. Then prove that there are no additional points.

26. In an affine plane geometry with n points on a line, how many lines total are there? Prove it!

27. *Prove that, if each line has exactly three points, there is only one structure that satisfies all the axioms of affine plane geometry, and this structure will consist of 9 points and 12 lines. See the figure below.



28. *Draw an affine plane geometry with only two points on a line. Is there more than one way to to this? Or just one?

29. *Is there an affine plane geometry with four points on a line?

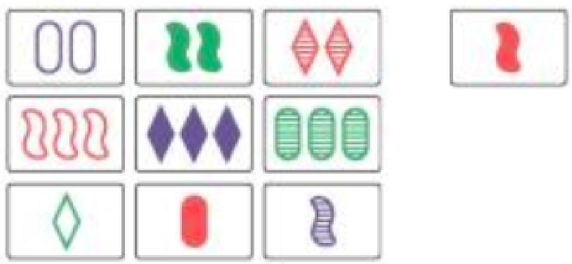
Counting Planes and Hyperplanes

30. Given three cards that form a SET (a line), how many different planes are there that contain this line?

31. How many planes are there in the deck?

32. If a line contains 3 cards, and a plane contains 9, how many cards would your expect a hyperplane (a 3-d plane) to contain? Can you build one?

33. Complete the hyperplane.



34. How many planes are in a hyperplane? How many lines are in a hyperplane?

35. How many cards would be in a 4-d hyper-hyperplane?

36. How many lines are parallel to a given line in a hyperplane? How many lines are parallel to a given line in a 4-d hyperhyperplane?

37. *What is the maximum number of cards you can have without having a single SET among them?

38. What would the answers to some of these questions be if you played SET with 5 attributes instead of 5. For example, you could play with three decks, one of which had dots on all cards, one of which had stripes on all cards, and one of which was plain?