Derangements

Abstract

In this worksheet, we will investigate a combinatorial object called a derangement. These are interesting objects in their own right, however the tools we can use to study them are also worthwhile concepts for exploration. We will introduce the concept of permutations in the opening. Along the way, the Principle of Inclusion-Exclusion and recursive sequences will prove to be profitable tools for investigating derangements.

This worksheet is adapted from an earlier version by former UNC-Chapel Hill graduate student Paul Kruse.

0 Motivating Example

In a (possibly apocryphal) experiment, twelve people were chosen, collectively representing each of the twelve zodiac signs of western astrology.[1] An astrologer then received personality profiles for these twelve people, and the task was to assign each their correct astrological signs. In this particular experiment, the astrologer correctly identified *none* of the correct signs!

Ignore what this says about the validity of astrology or the expertise of this particular astrologer. How many different guesses were the astrologer have made? How many of these guesses would, like the astrologer's, include zero correct parings? Assuming one made such guesses completely randomly, what is the probability that a random guess would correctly identify the signs of *zero* of the twelve people chosen?

1 Derangements

In this section, we introduce the notion of a derangement and work to understand some interesting facts about them.

Let n is a positive integer, so $\{1, 2, \ldots, n\}$ is the set of all positive integers no greater than n. Consider a function $f: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Assume that f satisfies the following conditions:

[1] I.e., Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius, and Pisces.

1. For all $i, j \in \{1,2,\ldots,n\}$, $f(i) = f(j)$ implies $i = j$.
   Equivalently, if $i \neq j$, then $f(i) \neq f(j)$.

2. For every $k \in \{1,2,\ldots,n\}$, there exists some $i \in \{1,2,\ldots,n\}$ such that $f(i) = k$.

That is, f sends distinct values in the domain to distinct values in the target, and every element in the target gets "hit" by f. Any f satisfying 1–2 is called a *permutation* of $\{1,2,\ldots,n\}$. The set of all such permutations is denoted S_n.

Exercise 1.1. How many permutations are there for $n = 3$? How about $n = 4$? And $n = 5$? What about general n?

A *derangement* is a permutation for which $f(i) \neq i$ for each $i \in \{1,2,\ldots,n\}$; that is, f sends every element something other than itself.[2]

A useful way to represent a permutation is in the following manner: Start with an open left parenthesis (. Next, write 1. Following 1, write $f(1)$, then write $f(f(1))$. Continue iterating until you return to 1, then closing with a right parenthesis) . (Do not repeat 1 here, however.) Once this happens, start with a left parenthesis (and the next $i \in \{1,2,\ldots,n\}$ not already appearing in the first cycle. Continue this process until all elements are used. This is called *cycle notation*.

For a worked example, let $n = 5$ and define f by $f(1) = 3$, $f(2) = 5$, $f(3) = 4$, $f(4) = 1$, $f(5) = 2$. In cycle notation, f is denoted $(1\ 3\ 4)(2\ 5)$.

Exercise 1.2. Write the following permutations in cycle notation:

1. $n = 5$, $f(1) = 3$, $f(2) = 4$, $f(3) = 5$, $f(4) = 1$, $f(5) = 2$.

2. $n = 5$, $f(1) = 1$, $f(2) = 4$, $f(3) = 5$, $f(4) = 2$, $f(5) = 3$.

In practice, if the cycle notation of a permutation includes a single number in parenthe-

[2] Equivalently: for any function $f\colon X \to X$, a point $x \in X$ such that $f(x) = x$ is called a *fixed point* of f. Then a derangement in S_n is any function with *no* fixed points.

ses, this can be omitted; it is understood in this case the the permutation maps that element to itself. For example, for $n = 4$, the permutation $(1\ 3)(2)(4)$ can be rewritten as $(1\ 3)$. The empty cycle, denoted $()$ represents the permutation that fixes every object.

Exercise 1.3. Let σ be a permutation on n objects. If σ is a derangement, how many numbers appear in its cycle notation representation?

Exercise 1.4. For $n = 6$, which of the following are derangements?

1. $(1\ 2\ 4)(3\ 5)$
2. $(1\ 5\ 2\ 6\ 3\ 4)$
3. $(1\ 2)(3\ 4)(5\ 6)$
4. $(1\ 5)$

Now, we list out the possible permutations for a fixed integer n and determine which are derangements.

Exercise 1.5. List out all of the permutations on $n = 1$ object. In this list, is there a derangement?

Exercise 1.6. List out all of the permutations on $n = 2$ objects. In this list, which of the permutations are derangements?

Exercise 1.7. Repeat the process above for $n = 3, 4$.

2 Principle of Inclusion-Exclusion

We now consider a particular useful way of counting. To begin, define a *set* to be an unordered list of objects. Any duplicates in the list are removed so there are no repeats. The objects in the set are called *elements* of the set. Examples of sets are:

- Days of the week = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
- Primary Colors = {Red, Yellow, Blue}
- Even integers = $\{\dots, -4, -2, 0, 2, 4, \dots\}$

If A and B are given sets, the *union* $A \cup B$ of A and B is the set consisting of objects, each of which is an element of A or B (or both). The *intersection* $A \cap B$ of A and B is the set consisting of objects, each of which is an element of both A and B. We can extend the notion of union to more than two sets by setting the union of sets $A_1, A_2, \dots, A_k$ to be the set of objects, each of which is an element of one of the A_i. Similarly, the intersection of sets $A_1, A_2, \dots, A_k$ is the set consisting of objects each of which are in every A_i.

If A is a set, it is called *finite* if the number of objects in A is finite; it is infinite otherwise. We denote by $|A|$ the *cardinality* of A, that is, the total number of objects. If it is a finite number, then A is a finite set; if A is not finite, then we say A has infinite cardinality. The unique set with no elements in it, the empty set, is denoted by $\varnothing$ and has cardinality 0.

One question to ask is given two finite sets A and B, what is the value of $|A \cup B|$?

Exercise 2.1. Let A be the set consisting of positive even integers less than 10 and B the set of positive integers less than 6. What is $A \cup B$? How about $A \cap B$? Can you relate the values of $|A|$, $|B|$, $|A \cup B|$, and $|A \cap B|$?

Exercise 2.2. Repeat the previous exercise with A consisting of even positive integers less than 10, B consisting of odd positive integers less than 10.

Exercise 2.3. Conjecture a formula that expresses $|A \cup B|$ in terms of $|A|$, $|B|$, and $|A \cap B|$ for finite sets A and B.

Now, suppose instead of two sets we consider a third set C.

Exercise 2.4. Keep the sets A and B the same as in Exercise 2.1. Let C be the set consisting of positive even integers greater than or equal to 4 and less than 12. Determine the value of $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$. Can you relate these to $|A \cup B \cup C|$?

Exercise 2.5. Do the same as above with Exercise 2.2, with $C = \{1,2,3,4\}$.

Exercise 2.6. Conjecture a formula that expresses $|A \cup B \cup C|$ in terms of $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$. Hint: Consider drawing a Venn diagram with A, B, C represented as circles positioned such that all intersections are represented, and keep track of how many times an element is counted for each intersection.

Exercise 2.7. Try to conjecture a formula that expresses $|A_1 \cup \cdots \cup A_k|$ in terms of $|A_i|$, $|A_i \cap A_j|$, ..., $|A_{i_1} \cap \cdots A_{i_{k-1}}|$ and $|A_1 \cap \cdots \cap A_k|$.

3 Applying Inclusion-Exclusion to Derangements

For $n \geq 1$, denote by A_i the set of permutations of n objects that fix object i.

Exercise 3.1. Let $n = 3$. Determine $|A_i|$ for $i = 1,2,3$. Next, determine $|A_i \cap A_j|$ for $1 \leq i < j \leq 3$. Finally, determine $|A_1 \cap A_2 \cap A_3|$. Use this to determine the value of $|A_1 \cup A_2 \cup A_3|$. What are the permutations not included in $A_1 \cup A_2 \cup A_3$?

Exercise 3.2. Let $n = 4$. Repeat the above exercise. Look at the value of all possible $|A_i|$ for $1 \le i \le 4$, $|A_i \cap A_j|$ for $1 \le i < j \le 4$, $|A_i \cap A_j \cap A_k|$ for $1 \le i < j < k \le 4$, and $|A_1 \cap A_2 \cap A_3 \cap A_4|$. Which permutations are not included in $A_1 \cup A_2 \cup A_3 \cup A_4$? How many are there?

Exercise 3.3. For general $n > 0$, what is the size of $|A_i|$? What is the size of $|A_i \cap A_j|$ for $i \ne j$? What is the size of $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ for $1 \le i_1 < i_2 < \cdots < i_k \le n$ where $1 \le k \le n$?

Exercise 3.4. For general n, how many pairs (i, j) with $1 \le i < j \le n$ are there? How many triples (i, j, k) with $1 \le i < j < k \le n$ are there? How many $k-$tuples $(i_1, i_2, \ldots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$ are there?

Exercise 3.5. Using the previous two exercises, and your knowledge of the generalized inclusion-exclusion formula, try to conjecture what the value of $|A_1 \cup \cdots \cup A_n|$ is. This counts the number of permutations that fix at least one object. Subtracting this from the total number of permutations will give the number of derangements on n objects.

One can verify that for $n = 1,2,3,4$, the number of derangements computed in previous exercises agrees with the formula above. For later purposes, we also define the number of derangements on zero objects as 1. Moreover, the summation $\sum_{j=0}^{n}(-1)^j \frac{1}{j!}$ approaches e^{-1} for large n, so that there are in general $\frac{n!}{e}$ derangements of n objects.

4 Recursive formula

A *sequence* of real numbers is a set of real numbers indexed by a subset of the integers. In practice, it is generally indexed by the nonnegative integers, denoted by $\{a_n\}_{n=0} = \{a_0, a_1, a_2, \ldots\}$.

Examples of sequences are:

1. $1,1,1,1,1,1,\ldots$
2. $1,2,4,8,16,\ldots$
3. $1,1,2,3,5,8,\ldots$
4. $-1,1,-1,1,-1,\ldots$

In many cases, a formula for the sequence may be hard to write down but relating certain terms with previous terms can be an easier task. When one can write down a formula expressing an element in the sequence in terms of previous elements, and applying this formula consistently from a certain index, we call this a *recursive definition*. This requires setting some initial conditions to make the formula make sense.

In the first example above, setting $a_0 = 1$ as the initial condition, one can the express the sequence as $a_{i+1} = a_i$ for $i \geq 0$.

In the second example, setting $a_0 = 1$ as the initial condition, one can express subsequent terms as $a_{i+1} = 2a_i$ for $i \geq 0$.

In the third example, we require two terms to be defined for the initial conditions. Setting $a_0 = a_1 = 1$, the sequence then follows the recursive definition $a_{i+2} = a_i + a_{i+1}$ for $i \geq 0$.

In the fourth example, setting $a_0 = -1$, the recursive definition is then $a_{i+1} = -a_i$ for $i \geq 0$.

Exercise 4.1. Consider the sequence $1,3,7,15,31,\ldots$. Set $a_0 = 1$. Try to express each terms a_{i+1} as a multiple of the previous term plus a constant term, i.e. $a_{i+1} = C \cdot a_i + D$ for constants C, D.

Exercise 4.2. Consider the sequence $1,2,5,26,677,\ldots$. Try relating the square of each previous term with the next term to determine a recursive relation.

5 Applying recursion to derangements

We now apply the notion of recursive sequences to derangements. Set $\mathcal{D}_n$ to be the set of derangements in S_n, and let $D_n := |\mathcal{D}_n|$ denote the number of these derangements.[3] We seek to write the D_n in terms of D_{n-1} and D_{n-2}. To this end, define $\mathcal{R}_k$ to be the set of derangements where k is in the nth position,[4] and set $r_k = |\mathcal{R}_k|$.

Exercise 5.1. For each derangement, examine the nth position. What value must not be in that position? Argue that $\mathcal{D}_n = \bigcup_{i=1}^{n-1} \mathcal{R}_i := \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_{n-1}$.

Exercise 5.2. What is the value of r_n? For $n = 3,4$ determine the values of r_i for $1 \le i \le n-1$. Conjecture how $r_1, r_2, \ldots, r_{n-1}$ are related.

Exercise 5.3. Is it possible for a derangement to be an element of both $\mathcal{R}_i$ and $\mathcal{R}_j$ for $i \neq j$? Express D_n in terms of all the r_i for $i = 1, \ldots, n-1$.

[3] The number of derangements in S_n is also denoted $!n$, so $!n = D_n = |\mathcal{D}_n|$.

[4] I.e., those derangements $\sigma \in \mathcal{D}_n$ such that $\sigma(k) = n$.

Examine $\mathcal{R}_{n-1}$ more closely. We can consider what these derangements look like. Recall that each of these place $n-1$ in the nth position.

Exercise 5.4. In $\mathcal{R}_{n-1}$, if n is placed in the $(n-1)$st position, what positions remain to be filled? Which numbers are available? Can you relate the positions and numbers available to a particular D_k for some k?

Exercise 5.5. In $\mathcal{R}_{n-1}$, consider the derangements in which k is placed in position $(n-1)$st position where $k \neq n, n-1$. If one swaps the positions of objects $n-1$ and n, and considers the restriction of this to objects $\{1, \dots, n-1\}$, this is now a permutation of $n-1$ objects. Show that this is a derangement in $\mathcal{D}_{n-1}$ and that the original derangement can be recovered. For example when $n = 5$, if the objects 1, 2, 3, 4, 5 are in positions $(3, 5, 1, 2, 4)$ then the new positions are given by $(3, 4, 1, 2)$ where objects 4 and 5 were swapped, and then object 5 forgotten.

In the previous two exercises, you showed that r_{n-1} can be written as the sum of two different D_k values.

Exercise 5.6. Try to use the previous exercises to write D_n in terms of multiples of different D_i terms for $i < n$.

From this expression, we can write a second more compact recurrence relation.

Exercise 5.7. Using this expression, show that $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$. Hint: It may be useful to recall that in a previous section, you determined that $D_i = i!\sum_{j=0}^{i}(-1)^j \frac{1}{j!}$ and observe that $\sum_{j=0}^{l-1}(-1)^j \frac{1}{j!} = \sum_{j=0}^{l}(-1)^j \frac{1}{j!} - (-1)^l \frac{1}{l!}$.