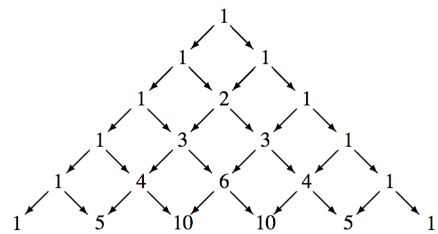
# Hidden Patterns in Pascal's Triangle*

# 1 Warm-Up



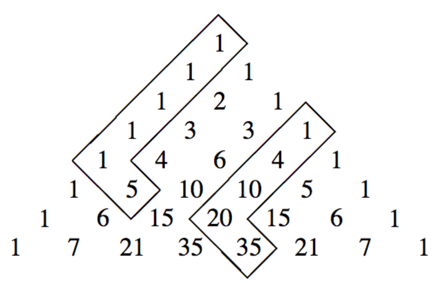

1. The figure above is rows 0 through 5 of Pascal's Triangle. Fill in a blank Pascal's triangle with lots more rows.

## 2 Patterns in Pascal's Triangle

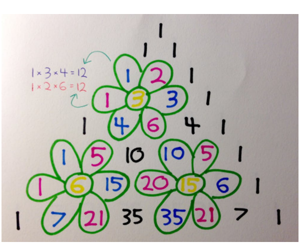
2. Add up all the numbers in each row. What pattern do you see in the sums? Why does this happen?

3. Next, alternately add and subtract the numbers in each row. For example, $1 - 4 + 6 - 4 + 1 = ?$. What pattern do you see? Why does this happen?

4. What do you notice about the numbers in the handle and the number in the tip of the hockey sticks below? Why does this work?



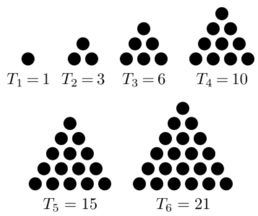

5. It is curious that the product of the six numbers surrounding any element in Pascal's Triangle is always a perfect square. Why does this work? Is there a connection between the number being surrounded and the value of the square?



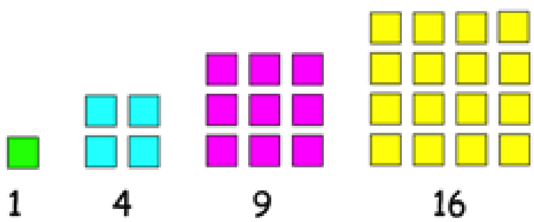

*From Tom Davis' www.geometer.org and *Pascal's Triangle* by Tony Colledge

6. See how many of the following famous sequences of numbers you can find in Pascal's triangle, either staring you in the face or hidden as the sum of some other numbers in Pascal's triangle?

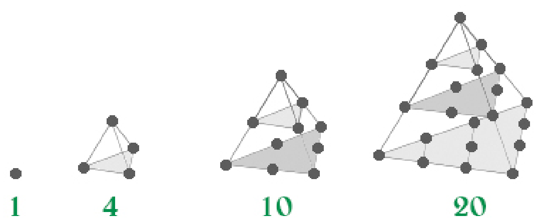
- Triangular numbers?



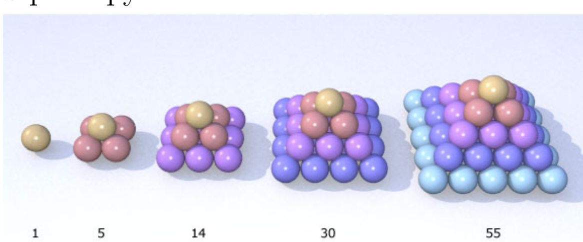

- Square numbers?




- Tetrahedral numbers?




- Square pyramid numbers




- Fibonacci numbers? (what is the rule for generating these numbers?)
  $1, 1, 2, 3, 5, 8, 13, 21, 35...$
- Powers of 11
  $1, 11, 11^2, 11^3, ...$

Can you explain why each of these sets of numbers appear where they do in Pascal's triangle?

## 3 Coloring Pascal's Triangle

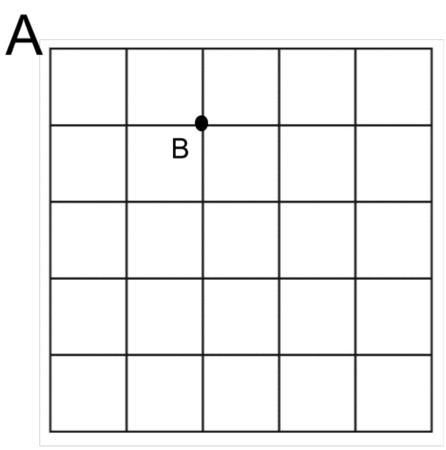
7. Take a large copy of Pascal's Triangle on graph paper with one square per number, starting at the upper left hand corner. Color each square black if it is an odd number and white if it is an even number. What visual patterns do you find? If you are having trouble fitting large numbers in squares, you could instead write out Pascal's triangle "mod 2" with 1's and 0's, where 1 is for an odd number and 0 is for even.

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | |
| 3 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | | |
| 4 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | | | |
| 5 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | |
| 6 | 1 | 0 | 1 | 0 | 0 | 0 | | | | | |
| 7 | 1 | 1 | 0 | 0 | 0 | | | | | | |
| 8 | 1 | 0 | 0 | 0 | | | | | | | |
| 9 | 1 | 1 | 1 | | | | | | | | |
| 10 | 1 | 0 | | | | | | | | | |
| 11 | 1 | | | | | | | | | | |

8. Try the same thing using Pascal's triangle "mod 3": color the squares black if they are not divisible by 3 and white if they are divisible by 3. Try mod 4, or mod 5, and mod 7.

## 4 Grid-walking and Coin-Flipping

9. A town has north-south streets and east-west streets. Suppose you start at the northwest tip $A$ and walk only south and east. At each intersection, write the number of ways you could get to that intersection. For example, there are three ways to get to intersection $B$.




What does this have to do with Pascal's Triangle?

10. If three coins are tossed, what is the chance of getting 0 heads? 1 head? 2 heads? 3 heads?

| Possible result for three coins | Possible ways of obtaining this result | # of ways | Probability |
|---|---|---|---|
| 0 heads, 3 tails | TTT | | |
| 1 head, 2 tails | TTH, THT, HTT | | |
| 2 heads, 1 tail | | | |
| 3 heads, 0 tails | | | |

What about 4 coins?

| Possible result for four coins | Possible ways of obtaining this result | # of ways | Probability |
|---|---|---|---|
| 0 heads, 4 tails | | | |
| 1 head, 3 tails | | | |
| 2 heads, 2 tails | | | |
| 3 heads, 1 tail | | | |
| 4 heads, 0 tails | | | |

Fill in the chart below with the probabilities for these numbers of coins.

| | 0 heads | 1 head | 2 heads | 3 heads | 4 heads | 5 heads |
|---|---|---|---|---|---|---|
| one coin | | | | | | |
| two coins | | | | | | |
| three coins | | | | | | |
| four coins | | | | | | |
| five coins | | | | | | |

How is this related to the previous problem about walking through town?

How can you use this coin example to show that the sum of the numbers on a row of pascal's triangle is a power of 2?

## 5 Choose Numbers (a.k.a. Binomial Coefficients)

11. The notation $\binom{5}{2}$ is pronounced "Five choose 2" and represents the number of ways to choose 2 kittens out of 5 to take home with you. The order in which you choose doesn't matter.

    (a) $\binom{5}{2}$ can be evaluated as $\dfrac{5 \cdot 4}{2 \cdot 1} = 10$. Explain why.

    (b) $\binom{7}{3}$ can be evaluated as $\dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$. Explain why.

12. (a) How can you evaluate $\binom{10}{3}$?

    (b) How can you find $\binom{10}{7}$?

    (c) How are these two numbers related?

    (d) How could you have predicted that relationship without doing any math?

13. Evaluate the numbers $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, $\binom{5}{3}$, $\binom{5}{4}$, $\binom{5}{5}$. Find these numbers in Pascal's Triangle.

14. Where does the number $\binom{n}{k}$ reside in Pascal's triangle?

15. Your friend's cat had 7 kittens and is giving them away. There is one especially playful black kitten that we'll call Spooky

    (a) In choose notation, write down the number of ways to choose 4 kittens to take home with you.

    (b) In choose notation, write down the number of ways to choose 4 kittens to take home with you if you insist on leaving Spooky behind.

    (c) In choose notation, write down the number of ways to choose 4 kittens to take home with you if you insist on taking Spooky home.

    (d) Write an equation to describe the relationship between these three numbers.

    (e) Use the kitten story to explain why the choose numbers can be found in Pascal's Triangle.

# 6 Extra: Finding formulas for sequences

16. Suppose you come across the sequence of numbers $5, 7, 21, 53, 109, 195, 317, 481, \ldots$ and you would like to find a general formula $f(n)$ such that $f(0) = 5$, $f(1) = 7$, $f(2) = 21$, $f(3) = 53$, and so on. One method that often works is to take differences of successive numbers and write them on the next line, then take differences of these differences, and write them on the next line, etc. See below.

$$\begin{array}{ccccccccccccccccc}
5 & & 7 & & 21 & & 53 & & 109 & & 195 & & 317 & & 481 & & \ldots \\
& 2 & & 14 & & 32 & & 56 & & 86 & & 122 & & 164 & & \ldots & \\
& & 12 & & 18 & & 24 & & 30 & & 36 & & 42 & & \ldots & & \\
& & & 6 & & 6 & & 6 & & 6 & & 6 & & \ldots & & & \\
& & & & 0 & & 0 & & 0 & & 0 & & \ldots & & & &
\end{array}$$

How can you use this method to guess the next 3 terms in the original sequence?

17. You can also get an explicit formula from this method. Consider the formula

$$f(n) = 5\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3}$$

Evaluate $f(0), f(1), f(2), f(3), f(4), f(5)$ and compare them to your original sequence.

18. Why does this formula work?

19. Use this method to find an explicit formula for $f(n) = 0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3$.

20. Make up your own sequence of numbers, trade with another student, and see if you can find formula's for each other's sequences.

# 7 And More

21. Look for other patterns in Pascal's Triangle. Try to figure out why these patterns work.