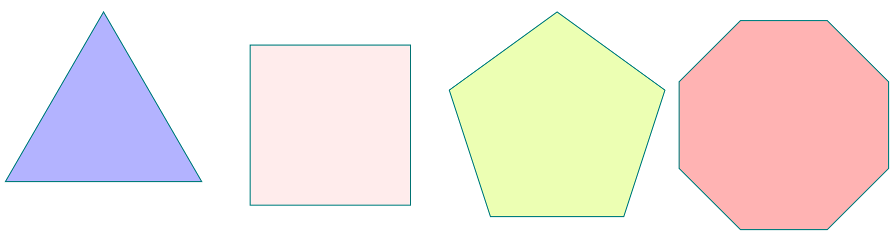
# Symmetry Part III: The Art of Counting [1]

Counting is one of the fundamental things that we, as humans, can do. While counting numbers may be an easy task, it's often extremely hard, if not impossible, to count the number of objects in certain collections. Counting is so important that mathematicians even have an entire field devoted to it, Combinatorics. It turns out that this field is at the heart of many other areas of study, including Calculus, Geometry, Topology, and, you guessed it, Symmetry.

Counting is truly an art, and it takes time and practice to develop the skills and insight needed for tricky problems. If these problems are challenging for you, that's a good thing!

## Warm Up

Recall from last time that we looked at the number of symmetries of regular polygons. Write down this number for each of the following polygons, and write down the two "main symmetries" we discussed.



## Problems Involving Symmetry

1. How many ways are there to put 6 different beads in order? $n$ different beads?
2. What if the beads are in a circle, so rotations don't matter?
3. What if the beads are on a bracelet, which can be rotated or flipped over?
4. Redo problem 1 through 3 with $w$ identical white beads and $b$ identical black beads. (Hint: Start with one white and one black, and work your way up. Can we solve it with 14 white and 6 black beads? Would it be much easier with 5 black instead of 6?)

   See how much harder it gets when there's symmetry?
5. How many ways are there to color the sides of a triangle with $k$ colors? (Four possible answers: with no symmetry, rotation only, reflection only, or rotation and reflection.)
6. How about an $n$-gon?

[1] Many of these problems are from *Mathematical Circles (Russian Experience). Others are from Paul Zeitz.*

7. How many ways are there to paint a six-sided die if you have one color of paint? Two colors? Three colors? $n$ colors? Note that painting the side with one dot red and the side with two dots blue is different from painting the side with one dot blue and the side with two dots red. You have to paint all the sides.

8. (Harder) Now try the above but for painting a cube with six identical faces.

## Additional Problems

9. We call a natural number ultra-odd if all of its digits are odd. How many four-digit ultra-odd numbers are there? How many four-digit ultra-even numbers are there (i.e. all digits even)?

10. You roll a 6-sided die 3 times. Among all possible outcomes, how many have at least one occurrence of the number 6?

**Permutations:**
Permutations: The number of ways to lay out $n$ different objects in a row is

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$$

.
**Combinations:**
The number of ways to choose $k$ objects out of a collection of $n$ objects is

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

11. Ms. Jewls' class is going to play capture the flag at recess. How many ways are there to divide the 28 students into 2 teams of 14? What if Maricia and Deedee can't be on the same team?

12. What is the largest number of triangles you can make by drawing 7 lines in the plane? The triangles may overlap or contain each other.

13. How many ways are there to rearrange the letters in the word "FLAMINGO" so that the vowels will be in alphabetical order? For example, GAFILMON (A - I - O).

14. How many ways are there to represent the number 12 as a sum of

    (a) 5 non-negative integers?

    (b) 5 positive integers?

    The order of the numbers matters here, so, for example, 1 + 4 + 5 + 1 + 1 is considered different from 1 + 1 + 1 + 4 + 5.

15. How many ways are there to choose 6 cards from a complete deck of 52 cards in such a way that all four suits will be present?