Thinking Combinatorially

Abstract

A number of algebraic identities arise from the branch of mathematics called *combinatorics*. In this session, we shall prove a number of algebraic identities by using combinatorial methods rather than other tools such as mathematical induction or more straightforward algebra.

0 Warmup Exercises

0.1 Let n be a nonnegative integer. What is $n!$, read as *n factorial*?

0.2 Let n, k be nonnegative integers. What is *n choose k*? Does your answer depend on whether k or n is the larger number?

Notation: n choose k is most typically denoted

$$\binom{n}{k},$$

with other notation like C_k^n, ${}_nC_k$, or $C(n,k)$ also common. The values $\binom{n}{k}$ are also called *binomial coefficients*.

0.3 The *multiplication principle*?[1] describes, among other things, how to compute $|S \times T|$, where $|S|$ and $|T|$ are finite sets. Explain this principle. Can you generalize?

[1]The multiplication principle is also called *the rule of product*, or *the fundamental principle of counting*.

0.4 What is *Pascal's Triangle*?[2]

1 Counting Subsets of Finite Sets

To begin, let us use the following convention to simplify notation:

Notation:. Let n be a positive integer. Then we set

$$[\mathbf{n}] := \{1, 2, \dots, n\}. \tag{1}$$

The following result is absolutely fundamental to combinatorics:

Theorem 1.1. *Let n, k be nonnegative integers with $k \leq n$. By definition the number of distinct k-element subsets of an n-element set is given by $\binom{n}{k}$, and*

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} \tag{2}$$

You are encouraged to try to *prove* Theorem 1.1, but our primary focus for now is being able to use this conceptual understanding of what $\binom{n}{k}$ *represents* in order to prove combinatorial identities.

Example 1.2. Consider the case $n := 4$, $k := 2$. Then by Theorem 1.1, there are precisely $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$ subsets of a 4-element set that contain precisely 2 elements.

For example, if $S := \{1, 2, 3, 4\}$, then the 2-element subsets of S are precisely $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$. We therefore see there are indeed 6 such 2-element subsets.

One of the ways Theorem 1.1 becomes especially useful is through the following:

Strategy 1.3. *Algebraic identities can often be verified by counting a particular set in two different ways.*

Example 1.4 (Pascal's Rule)**.** Let n, k be nonnegative integers. Then

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{3}$$

[2] "Pascal's Triangle" is the most common name for this object in English-language. Recognition of the triangle predates Pascal by centuries, though, and it is also known as the *Khayyam Triangle* (in Iran), *Yang Hui's Triangle* (China), and *Tartaglia's Triangle* (Italy).

Remark. Compare Example #1.4 with Warmup Exercise #0.4. What can we conclude?

We could prove (3) a number of algebraic ways. For example, one could use mathematical induction (say, inducting on n, leaving k fixed). Alternatively, one could verify (3) holds by replacing the values on both sides with the formula in (2). Instead, we illustrate Strategy 1.3.

Proof. Viewing Pascal's Rule conceptually, the claim is that the number of k-element subsets of an $(n+1)$-element set is equal to the sum of the number of k-element subsets of an n-element set and the number of $(k-1)$-element subsets of an n-element set.

Without loss of generality, set $S := [\mathbf{n+1}]$, so $|S| = n+1$. Consider any subset T of S having precisely k elements. There are two possibilities:

Case 1: $n+1 \notin T$.

Then since $n+1 \notin T$, T is thus a k-element subset of $[\mathbf{n}]$. Any such T is therefore a k-element subset of the n-element set $[\mathbf{n}] = S \setminus \{n+1\}$, and by Theorem 1.1, there are precisely $\binom{n}{k}$ such sets of this form.

Case 2: $n+1 \in T$.

Since $n+1 \in T$, we have that T is of the form $T' \cup \{n+1\}$, where T' is a $(k-1)$-element subset of $[\mathbf{n}]$. As in Case 1:, there are $\binom{n}{k-1}$ such subsets T', and thus $\binom{n}{k-1}$ such subsets T of S for which $n+1 \in T$.

The total number of k-element subsets of S is therefore the sum from these two cases, whence

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

which verifies (3). □

By Pascal's Rule, we see that the kth element in the nth row of Pascal's Triangle, beginning at the top with row $n = 0$, is $\binom{n}{k}$.

1.1 Let m, n be positive integers. Then

$$\binom{n+m}{2} = \binom{n}{2} + \binom{m}{2} + mn. \tag{4}$$

1.2 Let n be a nonnegative integer. Prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n. \tag{5}$$

1.3 Let n, m, and k be nonnegative integers such that $k \leq m \leq n$. Then

$$\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}. \tag{6}$$

1.4 Let n, k be nonnegative integers with $1 \leq k \leq n$. Then

$$k\cdot\binom{n}{k}=n\cdot\binom{n-1}{k-1}. \tag{7}$$

1.5 Let n be a positive integer. Prove that

$$\binom{n}{1}+2\cdot\binom{n}{2}+3\cdot\binom{n}{3}+\cdots+n\cdot\binom{n}{n}=n\cdot 2^{n-1}. \tag{8}$$

1.6 Prove *Vandermonde's Identity*: If m, n, and k are positve integers with $k \le m+n$, then

$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}$$
$$= \sum_{j=1}^{k} \binom{m}{j}\binom{n}{k-j}. \tag{9}$$

2 Counting with Bijections

Definition 2.1. Let $f: S \to T$ be a function between sets S and T. We say that f is a *bijection* (or a *one-to-one correspondence*) if and only if

- f is an *injection* or *one-to-one*: if $f(s) = f(s')$, then $s = s'$; equivalently, if $s \ne s'$, then $f(s) \ne f(s')$

  and

- f is *surjective* or onto: for every $t \in T$, there exists some $s \in S$ such that $f(s) = t$.

That is, every element in T gets touched by f, and by precisely one element in S.

It should be intuitively clear that two finite sets S and T are the same size if and only if[3] there is a bijection $f: S \to T$. We can therefore reformulate Strategy 1.3 as follows:

Strategy 2.2. *Let S, T be finite sets. To show that $|S| = |T|$, it suffices to produce a bijection $f: S \to T$.*

That is, a bijection effectively is *a way of counting a finite set in two different ways!*

2.1 Let n, k be nonnegative integers with $0 \le k \le n$. Then

$$\binom{n}{k} = \binom{n}{n-k}.$$

[3]For technical reasons, this remains true even when $S = T = \varnothing$.

2.2 Let n be a positive integer. Then

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

2.3 Let n be a positive integer. Then

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}. \tag{10}$$

2.4

3 Pascal's Triangle and The Binomial Theorem

Let x, y be indeterminates. If n is a positive integer, what can we say about $(x+y)^n$?

3.1 Expand $(x+y)^3$ and $(x+y)^5$. What can you say about the coefficients in these expansions?

3.2 Prove *The Binomial Theorem*:

Theorem 3.1 (The Binomial Theorem)**.** *Let x, y be indeterminates, and let n be a positive integer. Then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{0}y^n. \tag{11}$$

3.3 Prove Exercise #1.2 using The Binomial Theorem.

3.4 Prove Exercise #2.3 using The Binomial Theorem.

4 Additional Exercises

4.1 Let n, m, k be nonnegative integers with $m, k, \leq n$ and $k \leq n-m$. Then

$$\binom{n}{m} \cdot \binom{n-m}{k} = \binom{n}{k} \cdot \binom{n-k}{m}.$$

4.2 Let n, m, k be nonnegative inters with $k \leq m \leq n$. Without using Exercise #4.1, prove

$$\binom{n}{m} \cdot \binom{m}{k} = \binom{n}{k} \cdot \binom{n-k}{m-k}.$$

4.3 Let n, m, k be nonnegative integers with $k \leq m+n$. Then

$$\sum_{j=0}^{k} \binom{m}{j} \cdot \binom{n}{k-j} = \binom{m+n}{k}.$$

4.4 Let n, m, k be nonnegative integers with $k \leq n$. Then

$$\sum_{j=0}^{k} \binom{m}{j} \cdot \binom{n}{k+j} = \binom{m+n}{m+k}.$$

4.5 Let n, m be nonegative integers with $m \leq n$. Prove the *Hockey-Stick Identity*:[4]

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}. \quad (12)$$

[4] This gets its athletic name by locating each of the above elements in Pascal's Triangle. The resulting shape resembles a hocket stick.

Note: recall that by our definition of $\binom{n}{k}$, $mk = 0$ when $m < k$.

4.6 Let n be a nonnegative integer. Then

$$n^2 = 2 \cdot \binom{n}{2} + n.$$

4.7 Let n be a nonnegative integer. Then

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2 \cdot \binom{2n}{n} + \binom{2n}{n-1}.$$

4.8 Let n be a positive integer. Then

$$1 + 2 + 3 + \cdots + n = \binom{n+1}{2}. \tag{13}$$

4.9 Let n be a positve integer. Then

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}. \tag{14}$$

4.10 "A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committe could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N."

(2020 AIME I, Problem #7.)

References

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