Polytopes in 4 Dimensions

A *polytope* is a polyhedron in any dimension, not necessarily dimension 3.

- A 4-dimensional polytope has vertices, edges, faces and 3-dimensional "hyperfaces" (V, E, F, H).
- A 5-dimensional polytope has vertices, edges, faces, hyperfaces, and 4-dimensional "spaces" (V, E, F, H, S).

1 Warm-up Problems

1. Last time we saw that, according to a table, there are supposed to be 2 different 3-d polyhedra with 6 faces, 12 edges, and 8 vertices. The cube is one of them. What is the other?

2. A hypercube is a 4-dimensional cube. What are V, E, F, and H for a hypercube?

2 4-Dimensional Platonic Solids

3. How would you define Platonic solids in 4-dimensions? Can you give some examples?

4. Here are some vital statistics for the 4-d Platonic solids.

V	E	F	H
5	10	10	5
16	32	24	8
8	24	32	16
24	96	96	24
600	1200	720	120
120	720	1200	600

They are called the 4-simplex, the hypercube, the 4-orthoplex, the 24-cell (or octaplex), the 120-cell, and the 600-cell.

What patterns do you notice?

5. Find V, E, F, H for a pyramid over a 3-dimensional tetrahedron, cube, icosahedron, and dodecahedron.

6. Find V, E, F, H for a bipyramid over an octahedron.

7. What about V, E, F, H for a prism over a cube? Think about why this is the same thing as a square“times” a square.

8. What are V, E, V, H for a pentagon “times” a pentagon. Can you find formulas for V, E, F, H for the product of an m-gon and an n-gon?

And Beyond

9. What Platonic solids can you describe in 5 dimensions?

10. Calculate V, E, F, H, S for a 5-dimensional polytope, where S is the number of 4-dimensional spaces, and use these numbers to find the Euler characteristic $V - E + F - H + S$.

11. Which polytopes generalize easily to every dimension? What is Euler's formula in dimension n?

Hands on

12. Build models of the 4-dimensional Platonic solids. You will actually be building their *projections*, or shadows, in 3-dimensions.

    There are some instructions for building various models of the hypercube and 4-simplex and the 120 cell out of Zome in the book Zome Geometry.

Thanks to Matt Kahle for most of these problems.

3 Teacher Notes and answers

1. Last time we saw that, according to a table, there are supposed to be 2 different 3-d polyhedra with 6 faces, 12 edges, and 8 vertices. The cube is one of them. What is the other?

   Two pentagons attached at one edge like a hinge, two triangles jammed in each side of the hinge, and two squares completing the figure. Or, you can think of this as a tetrahedron with two vertices "truncated".

2. A hypercube is a 4-dimensional cube. What are V, E, F, and H for a hypercube?

   $V = 8$, $E = 24$, $F = 32$, $H = 16$

4 4-Dimensional Platonic Solids

3. How would you define Platonic solids in 4-dimensions? Can you give some examples?

   A polytope whose cells are all the same 3-d Platonic solids and whose edges have the same number of polyhedra around them and whose vertices all have the same number of polyhedra around them.

4. These are some vital statistics for the 4-d Platonic solids.

V	E	F	H
5	10	10	5
16	32	24	8
8	24	32	16
24	96	96	24
600	1200	720	120
120	720	1200	600

   They are called the 4-simplex, the hypercube, the 4-orthoplex, the 24-cell (or octaplex), the 120-cell, and the 600-cell.

   What patterns do you notice?

   $V - E + F - H = 0$. Also, some of these polytopes are duals of each other. Some are self-dual.

5. Find V, E, F, H for a pyramid over a 3-dimensional tetrahedron, cube, icosahedron, and dodecahedron.

   These are sort of like Pascal's triangle on the 3-d chart: put 1's at the ends of the 3-d row and add adjacent numbers.
   Pyramid over 3-d tetrahedron: $V = 5$, $E = 10$, $F = 10$, $H = 5$
   Pyramid over 3-d cube: $V = 9$, $E = 20$, $F = 18$, $H = 7$
   Pyramid over octahedron: $V = 7$, $E = 18$, $F = 20$, $H = 9$
   Pyramid over dodecahedron: $V = 21$, $E = 50$, $F = 42$, $H = 13$

Pyramid over icosahedron: $V = 13$, $E = 42$, $F = 50$, $H = 21$

6. Find V, E, F, H for a bipyramid over an octahedron.

   $V = 8$, $E = 24$, $F = 32$, $H = 16$. This is the 4-orthoplex!

7. What about V, E, F, H for a prism over a cube? Think about why this is the same thing as a square“times” a square.

   This is the hypercube: $V = 8$, $E = 24$, $F = 32$, $H = 16$

   You can make a chart of the numbers for the square on the column and the square on the row and put their products in the chart and add along the diagonals.

8. What are V, E, V, H for a pentagon “times” a pentagon. Can you find formulas for V, E, F, H for the product of an m-gon and an n-gon?

   Make a similar chart: get $V = 35$, $E = 50$, $F = 35$, $H = 10$

And Beyond

9. What Platonic solids can you describe in 5 dimensions?

   5-simplex (generalize tetrahedron, or make pyramid over 4-simplex), 5-hypercube, 5-orthoplex. There are no others!

10. Calculuate V, E, F, H, S for a 5-dimensional polytope, where S is the number of 4-dimensional spaces, and use these numbers to find the Euler characteristic $V - E + F - H + S$.

    Odd dimensions: add to 2, even dimensions: add to 0.

11. Which polytopes generalize easily to every dimension? What is Euler’s formula in dimension n?

    Again: n-simplex (generalize tetrahedron), n-hypercube, n-orthoplex. There are no others!
    Again: Odd dimensions: add to 2, even dimensions: add to 0.

Hands on

12. Build models of the 4-dimensional Platonic solids. You will actually be building their *projections*, or shadows, in 3-dimensions.

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