# 1 Warm-up

1. For two numbers $A$ and $B$, we say that $A \equiv B \pmod{12}$ if $A$ and $B$ have the same remainder when divided by 12.

   For example, $15 \equiv 3 \pmod{12}$.

   Fill in the blank with a small number: $\_\_\_\_\_ \equiv 39 \pmod{12}$?

2. For two numbers $A$ and $B$, we say that $A \equiv B \pmod{5}$ if $A$ and $B$ have the same remainder when divided by 5.

   - $8 \equiv 23 \pmod{5}$. Why?
   - $8 \not\equiv 14 \pmod{5}$. Why not?

3. $A \equiv B \pmod{10}$ means ...

   Fill in the blank with a small number: $577 \equiv \_\_\_\_\_ \pmod{10}$.

   (a) Is $13 \equiv 6 \pmod{5}$?

   (b) Is $85 \equiv 0 \pmod{5}$?

   (c) Is $17 \equiv 3 \pmod{7}$?

   (d) Is $5 \equiv 2 \pmod{4}$?

   (e) Is $4 \equiv -1 \pmod{5}$?

## 2 Mod n Trees

4. Fill in the blanks with the smallest positive numbers possible.

   (a) $52 \equiv \underline{\hspace{2em}} \pmod{12}$

   (b) $76 \equiv \underline{\hspace{2em}} \pmod{60}$

   (c) $15 \equiv \underline{\hspace{2em}} \pmod{7}$

   (d) $15 \equiv \underline{\hspace{2em}} \pmod{3}$

   (e) $15 \equiv \underline{\hspace{2em}} \pmod{11}$

   (f) $4588 \equiv \underline{\hspace{2em}} \pmod{10}$

   (g) $-7 \equiv \underline{\hspace{2em}} \pmod{10}$

5. Here is a drawing of the world (mod 3). On the tree with a 0 on the trunk, we put all the numbers that are congruent to 0 (mod 3). On the tree that with a 1 on the trunk, we put all the numbers that are congruent to 1 (mod 3). Write at least four numbers on each of the trees.

## 3 Adding and Multiplying (mod n)

6. Match the arithmetic problems on the left and the right that give the same answers.

| | |
|---|---|
| $10 + 15 \pmod 7$ | $5 + 1 \pmod 7$ |
| $12 + 22 \pmod 7$ | $10 \pmod 7 + 15 \pmod 7$ |
| $15 \times 22 \pmod 7$ | $15 \pmod 7 \times 22 \pmod 7$ |
| $14 \times 144 \pmod 7$ | $1 \pmod 7 \times 1 \pmod 7$ |
| | $0 \times 4 \pmod 7$ |

7. Compute these sums. Hint: you don't need to do a lot of arithmetic.
   (a) $423 + 577 \pmod{10}$
   (b) $56 + 89 \pmod{10}$
   (c) $892 + 9823 \pmod 5$
   (d) $901 + 723 \pmod 3$

8. Compute these products. Hint: be lazy.
   (a) $4893 \times 49024 \pmod{10}$
   (b) $3982734 \times 2398739 \pmod{10}$
   (c) $78 \times 23 \pmod 5$
   (d) $3874 \times 3284 \pmod 3$

## 4 Last digits

9. (a) What is the last digit of $14,306 + 908,797$? Can you find the answer quickly, without doing the whole addition problem?

   (b) What is the last digit of $5589 \times 4523$?

   (c) What is the last digit of $413 \times 5967 \times 4534$?

10. What is the last digit of $9^{99}$? Remember, $9^{99}$ means we multiply 9 by itself 99 times. Hint: try to find a pattern by finding the last digit of $9^1$, $9^2$, $9^3$, etc.

11. What is the last digit of $3^{2022}$?

12. What is the last digit of $2^{100}$?

**Challenge Problems:**

13. Find the remainder of $2^{100}$ when divided by 3.

14. Find the remainder when the number $3^{2022}$ is divided by 7.

15. Find the remainder when the number $9^{100}$ is divided by 8.

16. Find the remainder when the number $2019 \times 2020 \times 2021 + 2022^3$ is divided by 7

## 5 Square Numbers

1. Fill in the table to find the values of the square numbers mod 4. A square number is a number like 4 or 9 that is the square of another number.

| Number $X$ | Square Number $X^2$ | $X^2 \mod 4$ |
|---|---|---|
| 1 | 1 | |
| 2 | 4 | |
| 3 | 9 | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

2. Do you notice a pattern in the table above? What can you say about the square of an odd number mod 4? The square of an even number?
3. Is the number 114502909924083 a perfect square? Why or why not?
4. Is it possible to find two numbers, whose squares add up to 74?
5. Is it possible to find two numbers, whose squares add up to 1111? Hint: what is $1111 \mod 4$? What are the square numbers mod 4?
6. Can the sum of two square numbers be a square number?
7. Can the sum of squares of two odd numbers be a perfect square?
8. Can the sum of squares of three odd numbers be a perfect square?
9. Can the sum of squares of five consecutive numbers be a perfect square?

## 6 A Magic Trick

10. This is a magic trick performed by two magicians, A and B, with one regular, shuffled deck of 52 cards. A asks a member of the audience to randomly select 5 cards out of a deck. The audience member who we will refer to as C from here on then hands the 5 cards back to magician A. After looking at the 5 cards, A picks one of the 5 cards and gives it back to C. A then arranges the other four cards in some way, and gives those 4 cards face down, in a neat pile, to B. B looks at these 4 cards and then determines what card is in C's hand (the missing 5th card). How is this trick done?

    See https://math152.wordpress.com/2008/10/30/modular-arithmetic-and-a-cool-card-trick/ for an explanation.

**Extra Problems**

11. Start with 5 pieces of paper. At each step, choose one piece of paper and cut it into 4 pieces. Prove that you will never be able to get exactly 100 pieces of paper this way.

12. Is it possible to find a number $x$ so that both the following two facts are true?

    (a) $x \equiv 5 \mod 6$

    (b) $x \equiv 3 \mod 10$

13. Is it possible to find a number $x$ so that both of the the following two facts are true?

    (a) $x \equiv 7 \mod 9$

    (b) $x \equiv 5 \mod 12$