

## Topology of Surfaces

Most of this material is from *The Shape of Space* by Jeff Weeks. The word searches were taken from Torus Games at <http://geometrygames.org/TorusGames/index.html>. You can download the Torus Games app for many other games.

### 1 Warm Up: Geometry vs. Topology

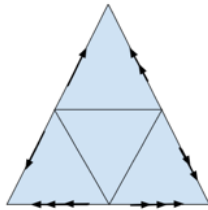
- The properties of an object that change when you bend or stretch it are the geometry of the object.
  - For example, angles and distances are parts of geometry but not topology.
- The properties of an object that stay the same when you bend or stretch it are called the topology of the object.
  - Two objects are considered the same topologically if you can deform one into the other without tearing, cutting, fusing, or other violent actions.

1. Which surfaces are topologically the same?

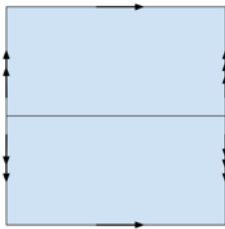


## 2 Gluing Diagrams

2. What topological surface do you get when you glue (or tape) the edges of the triangle together as shown?

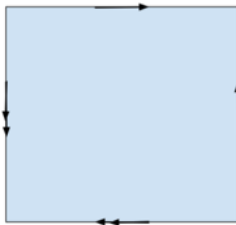


3. What do you get when you glue the edges of the square together like this?

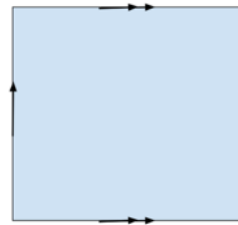


Don't glue the interior parts of the square together, just the edges!

4. What surface is this?



And this?

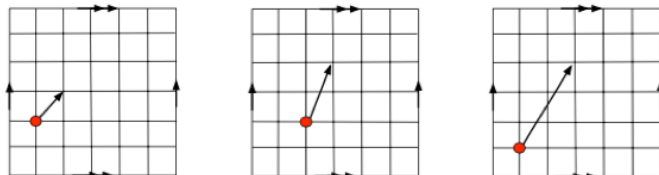


5. What happens as this 2-dimensional creature travels through its tiny universe?



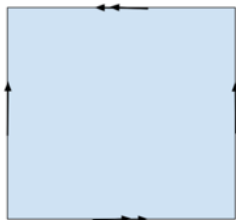
What does it see when it looks forwards? Backwards? Left? Right? (See Torus Games animation.)

6. A ladybug on a torus walks in a straight line until she returns to her starting point. Draw her path and find the length of her path if:
- she walks 1 unit northward for every 1 unit eastward, as shown at the left.
  - she walks 2 units northward for every 1 unit eastward as shown in the center.
  - she walks 3 units northward for every 2 units eastward as shown at right.



### 3 The Klein Bottle

7. What surface do you get when you glue together the sides of the square as shown?

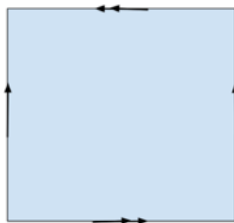


8. What happens as this creature travels through its Klein bottle universe?



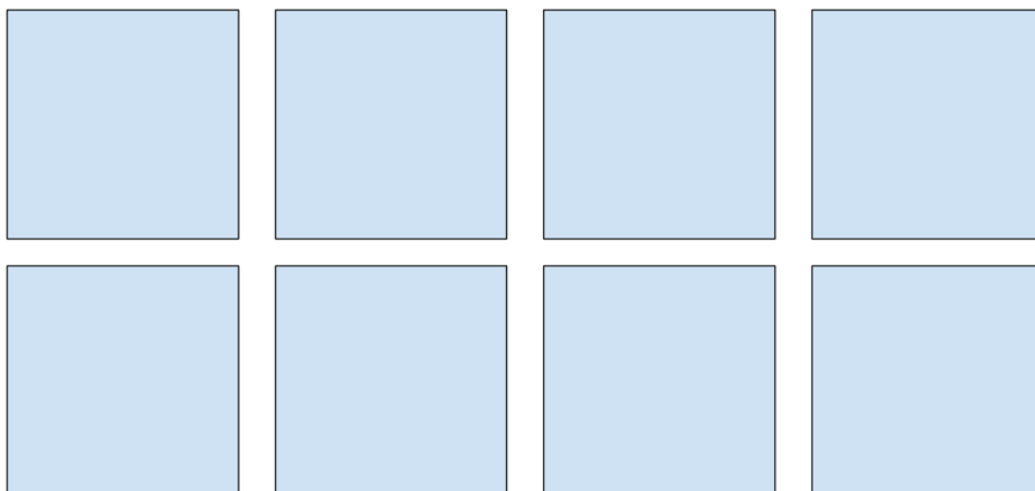
- A path that brings a traveler back to his starting point mirror-reversed is called an orientation-reversing path.
  - A surface that contains an orientation-reversing path is called non-orientable.
9. Can you find more than one orientation reversing path in this surface?

10. What happens when you cut a Klein bottle in half? Hint: It depends on how you cut it.



#### 4 Euler Characteristic

11. How many essentially different ways are there to glue the edges of a square in pairs? Draw the arrows below.

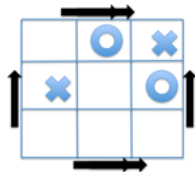


12. Which of the gluing diagrams that you drew above represent non-orientable surfaces?
13. For each gluing diagram that you drew, count up the number of faces, edges, and vertices AFTER gluing.
- When two edges are glued together, that only counts as one edge after gluing.
  - If several vertices end up glued together, that only counts as one vertex after gluing.

Which gluing diagrams have the same Euler characteristic and which have different ones?

### 5 Tic-Tac-Toe on a Torus

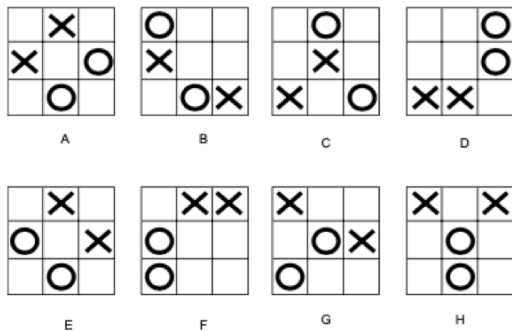
14. Where should X go to win? What if it's O's turn?



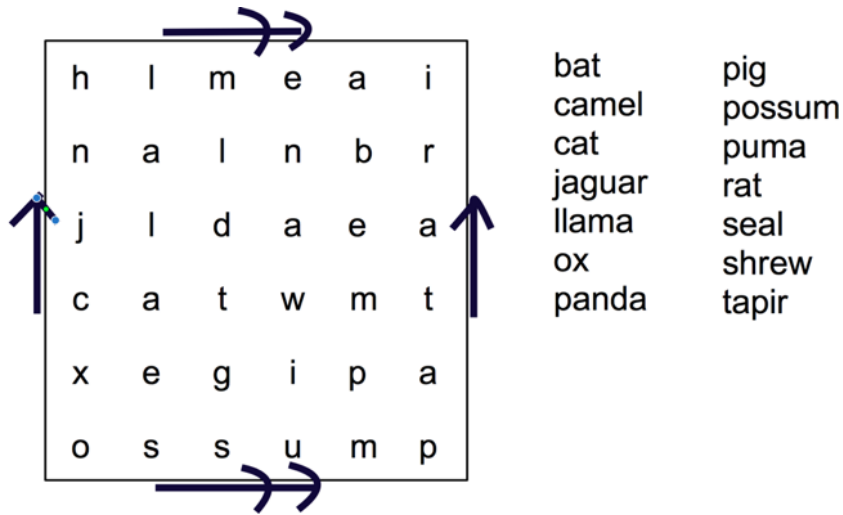
15. Play some games of Tic-Tac-Toe on the Torus with a classmate.

### 6 Torus Problems

16. Which of the following positions are equivalent in torus tic-tac-toe?

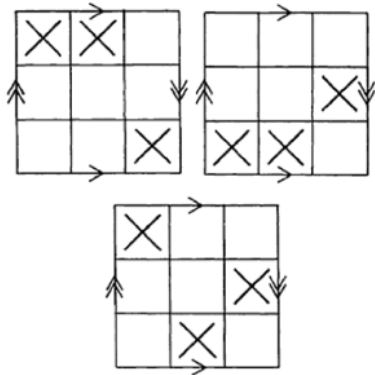


17. How many essentially different first moves are there in torus tic-tac-toe?
18. Is there a winning strategy for the first player in Tic Tac Toe on the torus? That is, is it possible for the first player to win no matter what the second player does?
19. A Cat's Game in Tic Tac Toe is a game where neither side wins, even though the board is filled up with X's and O's. Is it possible to have a Cat's Game in Tic Tac Toe on the torus?
20. Try this word search on the torus. Remember that the top and the bottom are glued together and the left and right sides are glued together.

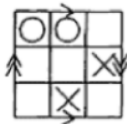


### 7 Tic-Tac-Toe on a Klein Bottle

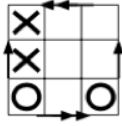
21. Which of these are winning positions in Klein bottle Tic-Tac-Toe?



22. Where can X go to win immediately in Klein bottle Tic Tac Toe?



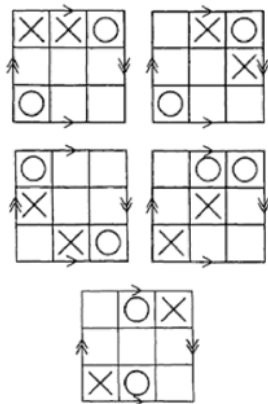
23. Where should X go to win?



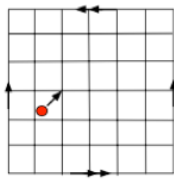
24. Play a few rounds of Klein bottle Tic-Tac-Toe with a classmate.

- Is there a winning strategy?
- Is it possible to get a cat's game?
- How many essentially different first moves are there?

25. What are the best moves for X in these positions?



26. A ladybug on a Klein bottle walks in a straight line until she returns to her starting point. She walks 1 unit northward for every 1 unit eastward. Draw her path.



27. Try this word search on the Klein bottle. The arrows show how the sides are glued together.



ash  
birch  
cedar  
elm  
fir  
larch

lilac  
maple  
oak  
palm  
pine  
poplar