CHMC Advanced Group:
Mathematical Games

Nov. 13, 2021

1 Introduction

In mathematics (and especially in the field of combinatorics), many sophisticated questions can often be phrased in the form of some combinatorial games often with simple-to-understand rules. Because of this, some often refer to Combinatorics as a field where "the questions are simple to ask but complex to answer" Today, we will spend some time playing some of these games, try to develop optimal playing strategies, and see if we can use mathematical structures to help us document our strategies.

2 1, 2, 3 Takeaway

For this game, start with a pile of 21 pennies between you and your partner. On each turn, a player may take either 1, 2, or 3 pennies from the pile, at which point the other player takes a turn. A player loses if there are no more legal moves to make on their turn (in this case, because there are no pennies remaining). Play a few rounds of this game with your partner before considering the questions below.

2.1 Questions

2.1.1. Are there any game positions (i.e. numbers of pennies at the start of your turn) that you know you can **always** find a winning move? Are there any game positions that you know will **never** have a winning move?

2.1.2. Does either player have an advantage in this situation? Is it better to be the first player to make a move or the second player?

2.1.3. What if instead, we start with a pile of 24 pennies? Then which player has an advantage? What if we start with n pennies?

2.2 Variations

Variation 1: 1, 2, 4 Takeaway: Play this same game, but now a player may take either 1, 2, or 4 pennies on their turn. How does this change this optimal strategy? Which player has an advantage whenever the game starts with n pennies?

Variation 2: 2, 3, 5 Takeaway: Play this same game, but now a player may take either 2, 3, or 5 pennies on their turn (you may not take just 1 penny, even if there is only 1 left). How does this change this optimal strategy? Which player has an advantage whenever the game starts with n pennies?

3 Two-Pile Nim

Now, we will start with two piles of pennies, one with 10 pennies and one with 7. On each turn a player may take any positive number of pennies they would like **but** they can only take from **one** of the two piles, at which point the other player takes a turn. A player loses if there are no more legal moves to make on there turn (i.e. if no pennies remain at the start of their turn). Play a few rounds of this game with your partner before considering the questions below.

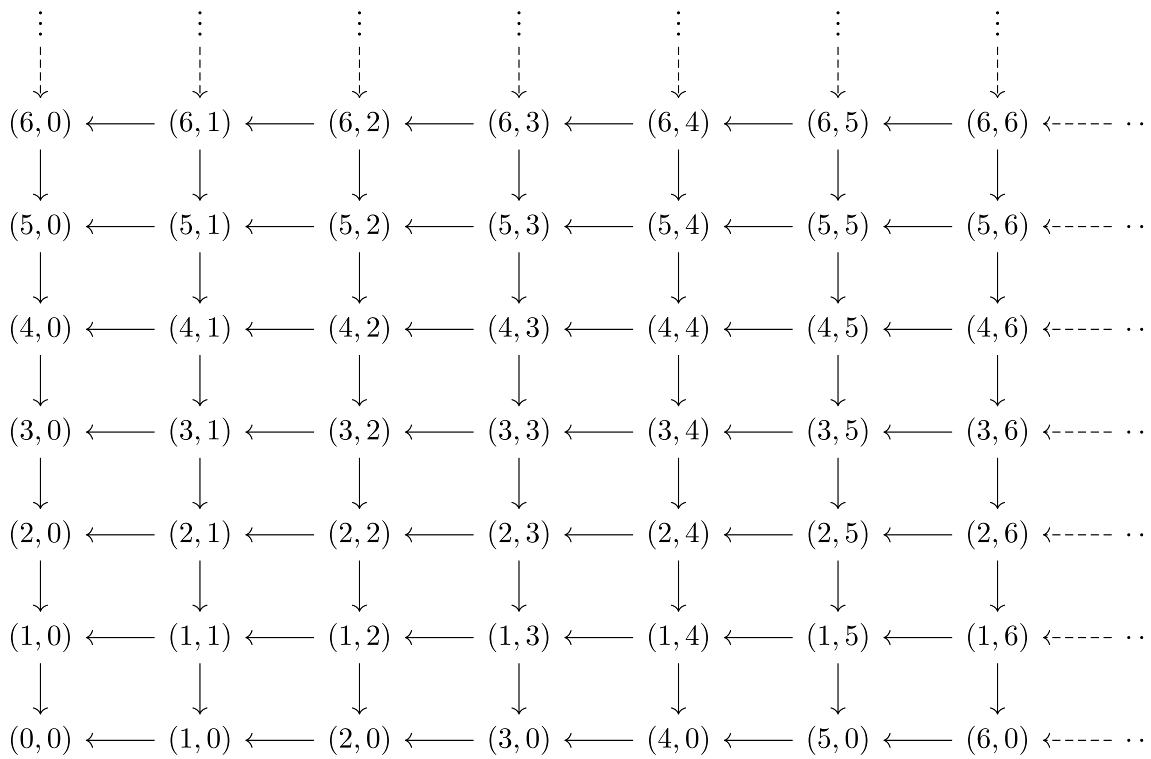
3.1 Questions

3.1.1. Are there any game positions that you know you can **always** find a winning move? Are there any game positions that you know will **never** have a winning move?

3.1.2. Does either player have an advantage in this situation? Is it better to be the first player to make a move or the second player?

3.1.3. What if instead, we start with a piles of 15 and 15 pennies? Then which player has an advantage? What if we start with piles of n and m pennies?

3.1.4. Use the diagram below to mark which game positions are safe (i.e. winning) positions and which are losing positions:



4 Puppies and Kittens

In this game, we start with two piles (called "Puppies" and "Kittens") just as in the last game; however, the set of legal moves is slightly different. This time on each turn, a player may take any number of pennies from the first pile, any number of pennies from the second pile, or they may take the **same** number of pennies from both piles. As before, a player loses if there are no pennies remaining at the start of their turn (i.e., whoever takes the last penny wins). Play a few rounds of this game with your partner before answering the questions on the next page.

4.1 Questions

4.1.1. How does this additional move change the optimal strategy you found in the last game? Why might that previous strategy not work anymore?

4.1.2. Use the diagram below to mark which game positions are safe (i.e. winning) positions and which are losing positions: