An Introduction to Mass Point Geometry, Part 2 of 2

Abstract

In this session, we shall continue our previous exploration of mass point geometry. Motivation comes from physics, generalizing the notion of the center of mass. Mass point geometry techniques can be powerful, offering much simpler ways to solve a number of problems from classical geometry, both in the plane and in higher dimensions. Our approach is modeled on that of [1] its subsequent adaptation in [2].

0 Review: Mass Point Geometry

Let's recall some of the basic definitions and results about mass point geometry which we introduced in our last session. Our approach is modeled on that of [1] its subsequent adaptation in [2].

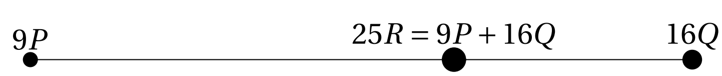
Definition 0.1. Let P be a point, and $m > 0$ a real number. Then a *mass point* is an expression denoted mP. Two mass points mP and nQ are by definition equal if and only if $m = n$ and $P = Q$. We shall often simply write "P" as shorthand for the mass point $1P$, too, provided the context is clear that this represents a mass point and not simply a point.

Definition 0.2. Let mP and nQ be two mass points. The *mass point sum* of mP and nQ, denoted $mP + nQ$, is defined as follows:

- If $P = Q$, then $mP + nQ := (m + n)P$.
- If $P \neq Q$, then $mP + nQ$ is defined to be mass point of mass $m + n$ at the unique point R on the line segment $\overline{PQ}$ that lies $n/(m + n)$ of the distance PQ from P to Q. That is,

$$\frac{PR}{RQ} = \frac{n}{m}.$$

- If mP is a mass point and $a > 0$, then we define $a(mP) := (am)P$.



Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

Proposition 0.3. *Let ℓO, mP, and nQ be mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative: $mP + nQ = nQ + mP$.*
- *Mass point addition is associative: $\ell O + (mP + nQ) = (\ell O + mP) + nQ$*

- *Mass point addition is distributive:* $a(mP+nQ) = amP+anQ$.

Proving both commutativity and distributivity is relatively straightforward. The associativity of mass point addition, however, is more complicated.

The following are strategies for using mass points:

0.1 Assume that $\overline{PQ}$ is a line segment containing the point R. Then if we know $PR/RQ = n/m$, assign masses m and n to P and Q respectively so that $mP+nQ = (m+n)R$. Further, we can scale this by a positive constant k to have $k(m+n)R = kmP+kmQ$, as well.

Recall that the numerator in this ratio is assigned to Q and the denominator is assigned to P, so that the *larger* mass is assigned to which of P or Q is *closer* to R.

0.2 Given mass points mP and nQ, use mass point operations to determine the location of R on $\overline{PQ}$ such that $mP+nQ = (m+n)R$.

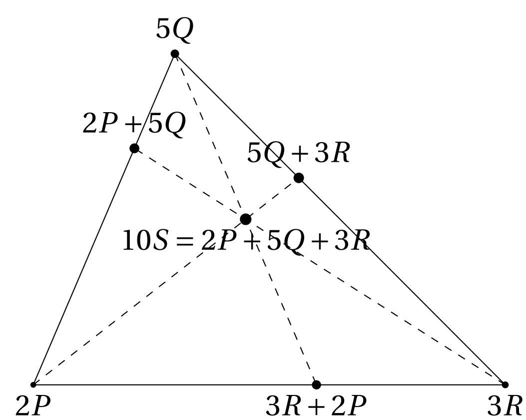
0.3 Let P, Q, and R be points. If for some masses m and n we have that $mP+nQ = (m+n)R$, then R must lie on segment $\overline{PQ}$.

This will be useful in showing two lines intersect in a particular point, or that three points are collinear.

0.4 We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.

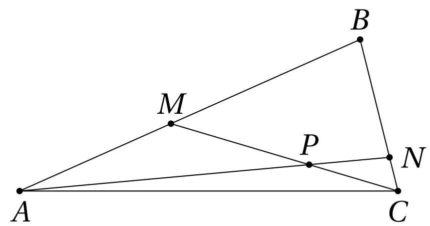
For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP+nP$.

As a reminder of how mass point operations are relevant for cevians in triangles, if P, Q, and R are the vertices of a triangle, then the mass point sum $2P+5Q+3R$ is indicated below:

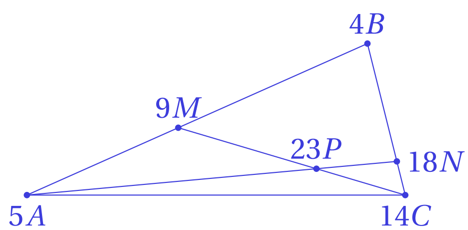


(As a reminder: a *cevian* of a triangle $\triangle ABC$ is a line segment, one of whose endpoints is a vertex of $\triangle ABC$, and where the other endpoint is any point of the opposite side, excluding the endpoints.)

Example 0.4. Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below.[1] If $AM/MB = 4/5$ and $BN/NC = 7/2$. Compute the ratios AP/PN and CP/PM.



Solution: Considering $AM/MB = 4/5$, consider the mass points $5A$ and $4B$. We also have $BN/NC = 2/7$. Since we have already assigned mass 4 to B, assign mass $4 \cdot \frac{7}{2} = 14$ to C so that $18N = (4+14)N = 4B + 14C$, whence $BN/NC = 14/4 = 7/2$, as desired. Then as above, $23P = 5A + 4B + 14C = 5A + (4B + 14C) = 5A + 18N$, so $\underline{AP/PN = 18/5}$. Similarly, $23P = (5A + 4B) + 14C = 9M + 14C = 5A + 18N$. (See below.) Therefore, $\underline{CP/PM = 9/14}$. □



1 Geometric Background: The Law of Sines, Angle Bisectors, and Areas

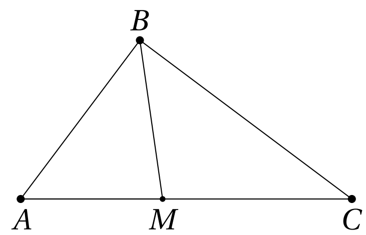
Let us review some preliminary results which will be useful in combination with techniques from mass point geometry. These can be proven, but they do not require mass point geometry.

Notation:. If $\triangle ABC$ is a triangle, then unless otherwise indicated, we let A, B, and C denote angles $\angle BAC$, $\angle ABC$, and $\angle ACB$, respectively. Further, we let a, b, and c denote the lengths BC, AC, and AB, respectively.

1.1 Let $\triangle ABC$ be a triangle. If M lies on $\overline{AC}$, then $\overline{BM}$ bisects $\angle ABC$ if and only if

$$\frac{c}{a} = \frac{AM}{MC}.$$

[1]This was Exercise #4.3 from the previous session



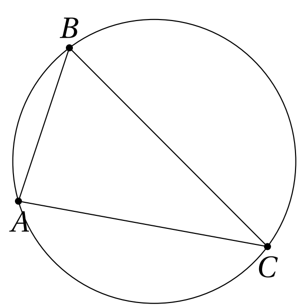
1.2 Let $\triangle ABC$ be a triangle, with M a point on $\overline{AC}$. Then

$$\frac{\text{Area}(\triangle ABM)}{\text{Area}(\triangle BMC)} = \frac{AM}{MC}.$$

That is, the ratio into which cevian $\overline{BM}$ divides the length of $\overline{AC}$ is the same as the ratio into which it divides the area of $\triangle ABC$ into smaller triangles $\triangle ABM$ and $\triangle BMC$.

1.3 *The Law of Sines:* Let $\triangle ABC$ be a triangle, where R is the radius of it circumcircle. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

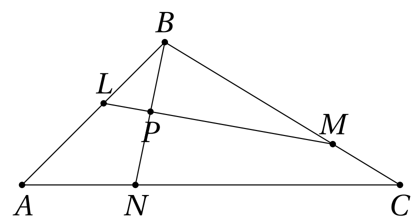


Remark. To clarify, a proof of this theorem need not use mass point geometry, though we may find The Law of Sines useful in future problems using mass point geometry.

Remark. For earlier exercises with mass point geometry, we assigned masses when we had information about the ratios of lengths of sides. If we have information about angles—or, equivalently, about their trigonometric values—then we may use that information to assign masses to points in such a way that relevant mass point sums balance at the endpoint of a cevian. See Exercises #2.3 and #2.4 for examples.

2 Exercises

2.1 Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point P, as below.[2] If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios LP/PM and BP/PN.



2.2 *Varignon's Theorem:* Let A, B, C, and D be the vertices in the plane of a (nondegenerate) quadrilateral. Let K, L, M, and N be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that $KLMN$ is a parallelogram. *Note:* we are not assuming that $ABCD$ is a convex quadrilateral.

2.3 Let $\triangle ABC$ be a triangle with cevian $\overline{BM}$ bisecting $\angle ABC$.

(a) Show that

$$\frac{AM}{MC} = \frac{\sin C}{\sin A}.$$

Equivalently, show that $AM \sin A = MC \sin C$.

[2] This was Exercise #5.1 from the last session.

(b) Assume $\sin A = 3/5$ and $\sin C = 7/25$. Consider the median $\overline{AN}$, which bisects $\overline{BC}$, and let P denote the intersection of $\overline{BM}$ and $\overline{AN}$. Compute AP/PN and BP/PM.

2.4 Let $\triangle ABC$ be a triangle. Show that the angle bisectors of $\triangle ABC$ are concurrent. That is, show that the angle bisectors intersect in a single point.

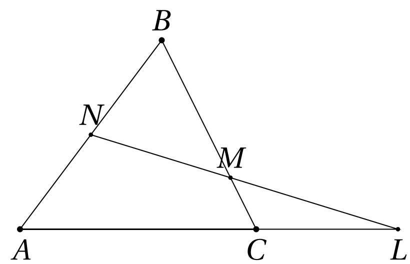
2.5 Let $\triangle ABC$ be an acute triangle. Prove that the altitudes of $\triangle ABC$ are concurrent.

2.6 *Ceva's Theorem:* Let $\triangle ABC$ be a triangle with cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$. Then the cevians are concurrent if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

Remark. Note that we could have solved Exercises #2.4 and #2.5 by first proving Ceva's Theorem, then showing that the product given above equals 1 in each case.

2.7 *Menelaus' Theorem:* Let $\triangle ABC$ be a triangle, L a point on the line given by $\overline{AC}$ as shown below, N any point in $\overline{AB}$, and M any point on $\overline{BC}$.



Then L, M, and N are collinear if and only if

$$\frac{AN}{NB} \cdot \frac{BM}{MC} \cdot \frac{CL}{LA} = 1.$$

2.8 In $\triangle ABC$, if cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at P, then

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

3 Exercises from Mathematics Competitions

3.1 (**from ARML 1989**) In $\triangle ABC$, angle bisectors $\overline{AD}$ and $\overline{BE}$ intersect at P. If the sides of the triangle are $a = 3$, $b = 5$ and $c = 7$, with $BP = x$ and $PE = y$, then compute the ratio x/y.

3.2 (**from AHSME 1975**) In $\triangle ABC$, M is the midpoint of side $\overline{BC}$, $AB = 12$, and $AC = 16$. Points E and F are taken on $\overline{AC}$ and $\overline{AB}$, respectively, and lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{AM}$ intersect at G. If $AE = 2AF$, find EG/GF.

3.3 (**from ARML 1992**) In $\triangle ABC$, points D and E are on $\overline{AB}$ and $\overline{AC}$, respectively. The angle bisector of $\angle A$ intersetcs $\overline{DE}$ at F and $\overline{BC}$ at T. If $AD = 1$, $DB = 3$, $AE = 2$, and $EC = 4$, compute the ratio AF/AT.

3.4 (**from AIME 1988**) Let P be an interior point of $\triangle ABC$, and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = 5$, $CP = c$, and let the extensions from P to the opposite sides all have length D. If $a + b + c = 43$ and $d = 3$, then find abc.

References

[1] Tom Rike, *Mass point geometry*, http://mathcircle.berkeley.edu/sites/default/files/archivedocs/2007_2008/lectures/0708lecturespdf/MassPointsBMC07.pdf, 2007, online: retrieved December 9, 2016.

[2] Zvezdelina Stankova, Tom Rike, and editors, *A decade of the Berkeley Math Circle: The American experience*, vol. I, Mathematical Sciences Research Institute and The American Mathematical Society, Providence, Rhode Island, USA, 2008.