An Introduction to Continued Fractions, Part 1 of 2

Abstract

In this session, we shall explore *continued fractions* and their applications. Our approach is modeled on that of The Ross Mathematics Program (formerly The Ross Young Scholars Program), as well as more traditional texts such as [1] and [2].

0 Continued Fractions: Basic Concepts and Notation

This week we explore the concept of *continued fractions*. These may first seem like artificial constructions, but they are incredibly useful in obtaining "good" rational approximations to real numbers, as well as solutions to at least two classes of Diophantine equations.

Definition 0.1. A *finite simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}},$$

where a_0 is an integer, and $a_1, a_2, \dots, a_n$ are all positive integers. To simplify this notation we let

$$[a_0; a_1, a_2, \dots, a_n]$$

denote the above expression.

Definition 0.2. An *infinite simple continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}},$$

where a_0 is an integer, and $a_1, a_2, \dots, a_n, \dots$ are all positive integers. As in Definition 0.1, we let

$$[a_0; a_1, a_2, \dots, a_n, \dots]$$

denote the above infinite simple continued fraction.

The "simple" in "simple continued fraction" refers to the fact that all the numerators in this expression are equal to 1. Since we shall be considering *only* simple continued fractions, we consider "simple" to be implicit in the discussion below.

Definition 0.3. Let $\alpha := [a_0; a_1, a_2, \ldots, a_n, \ldots]$ be an infinite simple continued fraction. We say that α is *periodic* if the sequence eventually repeats. That is, α is periodic if and only if it is of the form

$$[a_0; a_1, \ldots, a_k, a_{k+1}, \ldots, a_m, a_{k+1}, \ldots, a_m, \ldots].$$

We denote a periodic infinite continued fraction as above by

$$[a_0; a_1, \ldots, a_k, \dot{a}_{k+1}, \ldots, \dot{a}_m]$$

or

$$[a_0; a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_m}].$$

Definition 0.4. Let α be either a simple finite continued fraction or an infinite simple continued fraction as in Definitions 0.1 and 0.2. Further, let k be an integer with $0 \le k \le n$. Then the kth *convergent* to α is the finite continued fraction

$$\alpha_k := [a_0; a_1, \ldots, a_k].$$

Definition 0.5. Let α be either a finite or infinite simple continued fraction as above. Then the *magic table* for α is an array of the form

		a_0	a_1	a_2	a_3	a_4	a_5	$\cdots$
0	1	P_0	P_1	P_2	P_3	P_4	P_5	$\cdots$
1	0	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	$\cdots$

The P_j and Q_j are integers defined by the following recurrence relations:

$$\begin{aligned} P_{-2} &:= 0 & Q_{-2} &:= 1 \\ P_{-1} &:= 1 & Q_{-1} &:= 1 \\ P_0 &:= a_0 & Q_0 &:= 1 \\ P_k &:= a_k P_{k-1} + P_{k-2} & Q_k &:= a_k Q_{k-1} + Q_{k-2}. \end{aligned}$$

Example 0.6. Let $\alpha := [3; 1, 2, 4]$. Then the magic table for α is

		3	1	2	4
0	1	3	4	11	48
1	0	1	1	3	13

1 Introductory Exercises

Let's begin with some simple computational exercises about computations with continued fractions.

1.1 Simplify $[-1;7,2]$ as a fraction of the form P/Q, where P, Q are integers, $Q > 0$, and P/Q is in lowest terms.

1.2 Compute, as above, $[0;9,1,3]$.

1.3 Compute the continued fraction expansion for $\frac{29}{11}$. What is the magic table for this continued fraction expansion?

1.4 Compute each convergent for $\alpha := [4;2,1,3]$. Further, construct the magic table for α. *Note:* Compare the magic table for α to that given in Example #0.6 above, too.

1.5 Compute the infinite continued fraction for $\sqrt{2}$, and compute the first few entries of its magic table. Further, compute values of $P_k^2 - 2Q_k^2$ for the first few columns of the magic table.

1.6 Assuming convergence, compute the value for the infinite continued fraction $\alpha := [1;1,1,\dots] = [1;\dot{1}]$. Construct the magic table, and compute the first few entries.

1.7 Compute the infinite continued fraction expansion for $\sqrt{2}$. For the first few columns of its magic table, compute the values $P_k^2 - 41Q_k^2$.

2 Properties of Convergents and the Magic Table

Next, we consider some exercises exploring the properties of the convergents and the magic table for a given continued fraction. *Note:* proofs for many of these exercises can be obtained via mathematical induction. If you're unfamiliar with induction, then find a volunteer or fellow student who can explain it.

2.1 Prove that for any continued fraction $[a_0;a_1,a_2,\dots,a_n]$, we have $Q_0 \geq 1$, $Q_1 \geq 1$, $Q_2 \geq 2$, $Q_3 \geq 3$, $Q_4 \geq 5$, and in general, $Q_n > Q_{n-1}$ and $Q_n > n+1$ for all $n \geq 3$. (Can you provide an even better lower bound for Q_k?)

2.2 Let $[a_0;a_1,a_2,\dots,a_n]$ be a continued fraction. Prove that for each integer k with $1 \leq k \leq n$, we have

$$\det \begin{bmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{bmatrix} := P_{k-1}Q_k - Q_{k-1}P_k = (-1)^k.$$

2.3 Let $[a_0; a_1, a_2, \ldots, a_n]$ be a continued fraction. Prove that for each integer k with $1 \le k \le n$, we have

$$\det \begin{bmatrix} P_{k-2} & P_k \\ Q_{k-2} & Q_k \end{bmatrix} := P_{k-2}Q_k - Q_{k-2}P_k = (-1)^{k-1} a_k.$$

2.4 Let $\alpha := [a_0; a_1, \ldots, a_n]$ be a continued fraction. Prove that $\alpha = \frac{P_n}{Q_n}$; that is, prove that the continued fraction α is recovered as the quotient entries under index n in the magic table. Moreover, prove that $\frac{P_n}{Q_n}$ is already in lowest terms.

2.5 Let $\alpha := [a_0; a_1, \ldots, a_n]$, where each $a_k > 0$. If $\alpha = \frac{P_n}{Q_n}$, prove that $\alpha' := [a_n; a_{n-1}, \ldots, a_1, a_0] = \frac{P_n}{P_{n-1}}$. Compare to Example #0.6 and Exercise #1.4.

2.6 Prove that for all k for which the quantity makes sense,

$$\left| \frac{P_k}{Q_k} - \frac{P_{k+1}}{Q_{k+1}} \right| = \frac{1}{Q_k Q_{k+1}} \le \frac{1}{Q_k^2}.$$

2.7 Let $\alpha := [a_0; a_1, \ldots, a_n]$ be a finite continued fraction. Prove that for each k with $0 \le k \le n$ and $\alpha_k := [a_0; a_1, \ldots, a_k]$, we have

$$\alpha_0 < \alpha_2 < \alpha_4 < \cdots < \alpha < \cdots < \alpha_5 < \alpha_3 < \alpha_1.$$

2.8 Let $[a_0; a_1, a_2, \ldots, a_n, \ldots]$ be an infinite continued fraction, as in Definition def:infinite continued fraction. Explain why this infinite continued fraction must represent an actual real number. For those of you with some basic understanding of the relevant concepts, this means that you should give some argument why

$$\lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n]$$

exists.

2.9 Let α be any real number. Explain how to obtain a continued fraction representation for α. Must this continued fraction for α be unique?

3 Linear Diophantine Equations

3.1 Let a, b be positive integers. If $\gcd(a, b) = 1$, describe a method guaranteed to produce integers x, y such that $ax + by = 1$. More generally, if $\gcd(a, b) = d$, how can we produce integers x, y such that $ax + by = d$?

3.2 Let a, b be positive integers. If $d := \gcd(a, b)$, then describe a method to find integers x and y such that $ax + by = d$.

3.3 Let a, b, and c be arbitrarily given integers. Provide, with justification, a set of criteria for which the equation

$$ax + by = c$$

has a solution such that x and y are both integers.

3.4 Let a and m be integers, with $m > 1$. We say that x is the *multiplicative inverse of a modulo m* if and only if x is an integer and $ax \equiv 1 \pmod{m}$. (For those unfamiliar with modular arithmetic: this means that $ax - 1$ is divisible by m.) *Important:* note that x must itself be an integer!

Fix an integer $m > 1$. Give a complete characterization of all integers a such that a has a multiplicative inverse modulo m. For those a admitting a multiplicative inverse modulo m, provide a method for computing such a multiplicative inverse.

3.5 Compute, if possible, the following multiplicative inverses modulo m for a:

(a) $a := 4$, $m := 17$

(b) $a := 43$, $m := 257$

(c) $a := 17$, $m := 119$

3.6 Let $a_1, \ldots, a_k, c$ be integers. When can we solve the Diophantine equation

$$a_1 x_1 + a_2 x_2 + \cdots a_k x_k = c$$

for integers $x_1, \ldots, x_k$? For those cases where a solution exists, describe an algorithm for producing at least one solution.

4 Rational Approximations

4.1 Let α be an irrational real number. Then for all nonnegative integers n, we have

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2}.$$

Note: A priori, we could expect that

$$\left|\alpha - \frac{P_n}{Q_n}\right| \leq \frac{1}{2Q_n}.$$

Being able to approximate α to within the *square* of the denominator of the convergent is a considerable improvement over this initial worst-case scenario.

4.2 Let α be a real number, and let $\frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be consecutive convergents to α. Then

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \left|\alpha - \frac{P_{n-1}}{Q_{n-1}}\right|.$$

4.3 Here, we shall build up to a basic result establishing another sense in which the convergents to α are "good" rational approximations to α. First, we state the following without proof:

Theorem 4.1 (Pick's Theorem)**.** *Let P be a simple polygon in the plane, each of whose vertices is a lattice point. (That is, for each vertex V on the boundary of P, both the x- and y-coordinates of V are integers.) Let I denote the number of lattice points in the interior of P, and let B denote the number of lattice points on the boundary. (I.e., B counts the number of lattice points that lie on the edges of the boundary of P.) Then the area A of P is given by the formula*

$$A = I + \frac{B}{2} - 1.$$

(a) Let (a,b) and (c,d) be points in the plane. Consider the parallelogram P whose vertices are $(0,0)$, (a,b), (c,d), and $(a+c,b+d)$. Then

$$\text{Area}(P) = \left|\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right| = |ad = bc|.$$

(b) Let α be a real number, and let $\frac{P_k}{Q_k}$ be a convergent to α. If $\frac{P}{Q}$ is any rational number with $P, Q \in \mathbb{Z}$, $Q > 0$, and $Q < Q_k$, then

$$\left|\alpha - \frac{P_k}{Q_k}\right| < \left|\alpha - \frac{P}{Q}\right|.$$

That is, we cannot find a better rational approximation to α than $\frac{P_k}{Q_k}$ using a smaller denominator.

4.4 Let α be an irrational number. Prove that for infinitely many convergents $\frac{P_n}{Q_n}$, we have

$$\left|\frac{P_n}{Q_n} - \alpha\right| < \frac{1}{2Q_n^2}.$$

Hint: Prove that for any two consecutive convergents, at least one must satisfy this stronger condition.

Note: In general, the best possible result we can obtain is that there are infinitely many convergents such that

$$\left|\frac{P_n}{Q_n} - \alpha\right| < \frac{1}{\sqrt{5}Q_n^2};$$

the constant $\sqrt{5}$ is optimal if and only if α is of the form $\alpha = [a_0; a_1, \ldots, a_n, \dot{1}]$. Otherwise, $\sqrt{8}$ is the optimal constant if and only if $\alpha = [a_0; a_1, \ldots, a_n, \dot{2}]$. There are a number of other refinements possible.

5 To Be Continued...

In our next session, we will further explore continued fraction methods. In particular, we shall consider in more generality *Pell's Equation*, the Diophantine equation of the form $x^2 - dy^2 = 1$, where d is a positive integer.

References

[1] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, Walton Street, Oxford OX2 6DP, fifth edition, 1979.

[2] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery. *An Introduction to the Theory of Numbers.* John Wiley & Sons, Inc., New York, fifth edition, 1991.