Invariants [1]

1. A code uses only the letters A and O, and satisfies the following properties:

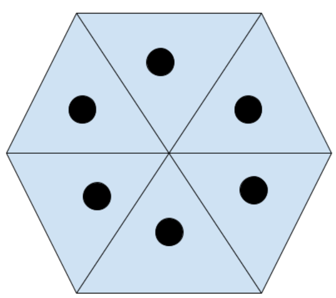
   (a) If you delete two neighboring letters AO from any code, or insert them, then you get a code with the same meaning.

   (b) If you delete two neighboring letters OA from any code, or insert them, then you get a code with the same meaning.

   (c) If you delete four neighboring letters AAOO from any code, or insert them, then you get a code with the same meaning.

   Do the codes AAO and OAA have the same meaning, or different meanings, or is there not enough information to be sure?
   What about the codes AAO and OOA?

2. A game is played on a circle divided into 6 sectors, as shown. To start, a stone is placed in each sector. On your turn, you can pick up any two stones, and move each stone to a sector that borders its original position. You win the game if you can move all the stones into a single sector. Give a winning strategy or show that it is not possible to win.



3. There are 7 plastic cups on a table, all standing upside down. You are allowed to turn over any 4 of them in one move. Is it possible to get all the cups standing up?

4. There are six sparrows sitting on six trees, one sparrow in each tree. The trees stand in a row, with 10 meters between any two neighboring trees. If a sparrow flies from one tree to another, then at the same time, some other

[1] Problems from Mathematical Circle (Russian Experience)

sparrow flies from some tree to another the same distance away, but in the opposite direction. Is it possible for all the sparrows to gather on one treee? What if there are seven trees and seven sparrows?

5. In an 8 x 8 table, one of the boxes is colored black and all the others are white. Prove that it is not possible to make all the boxes white by ”recoloring” the rows the columns. ”Recoloring” means changing the color of all the boxes in a single row or column.

6. Solve the same problem for a 3 x 3 table if initially there is only one black box in a corner of the table.

7. Solve the same problem for an 8 x 8 table if initially all four corner boxes are black and all the others are white.

8. There are 11 green, 13 brown, and 16 red chameleons on an island. When two chameleons of different colors meet, they both change their colors to the third color. For example, if a green and a red chameleon meet, they both become brown. Is it possible for all chameleons to be the same color?

9. Same problem, but on a different island where there are 13 green, 15 brown, and 17 red chameleons to start.

10. Dr. Gizmo has invented a coin changing machine which can be used in any country in the world. No matter what the system of coinage, the machine takes any coin, and, if possible, returns exactly 5 coins with the same total value. Prove that no matter how the coinage system works in a given country, you can never start with a single coin and end up with 26 coins.

11. A special chess piece called a camel moves along a 10 x 10 board like a (1, 3)-knight. That is, it moves to any adjacent square and then moves three squares in any perpendicular direction. Is it possible for a camel to go from some square to an adjacent square?

12. The numbers 1, 2, 3, ... 2021 are written on a blackboard. You are allowed to erase any two of them and replace them with their difference. Is it possible to get to a situation in which all the numbers are zeros?

13. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than 1 stone, throw away one stone from that heap, and divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones?