Infinity

1 Warm-Up: Hilbert's Grand Hotel

1. Hilbert's Grand Hotel has infinitely many rooms, numbered $1, 2, 3, 4, \ldots$.

   (a) One stormy night, every single room is full, when a tired, wet traveler shows up at the door, looking for a place to stay. You are working the night shift, and you don't want to turn the guest away, but you can't ask any guests to share rooms. How can you accommodate this new guest? What if the guest actually has five friends waiting outside and each requires a private room?

   (b) After managing to accommodate this traveler (and his five friends), you go back to doing your homework, when you are interrupted by another knock at the door. A bus full of infinitely many travelers (numbered $1, 2, 3, \ldots$) has arrived and each one needs a private room. Can you accommodate them?

   (c) Next, you find that infinitely many buses, with license plates numbered $1, 2, 3, \ldots$ have all pulled into the parking lot. Each bus has infinitely many passengers numbered $1, 2, 3 \ldots$ on it. Is there any hope at all of accommodating them?

2 Bijections

2. Which is bigger, the set of people in this room or the set of cups in my hand? How can you explain without counting them / without using numbers?

3. Here are some definitions.

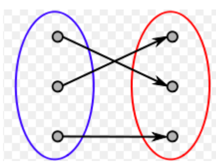
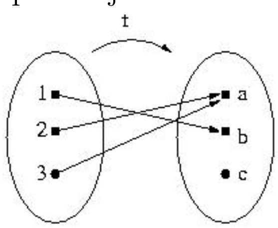
   - A *map* or *function* is a rule that takes input numbers (or things) to output numbers (or things).

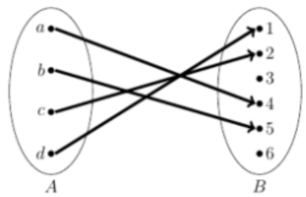
     For example, the map $x \to x + 1$ from the positive to the positive integers takes the number 2 to the number 3, the number 17 to the number 18, and every positive integer to the next higher integer.

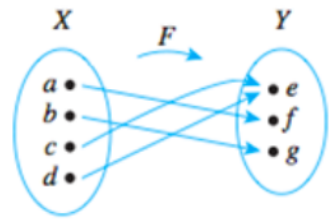
     Draw arrows to describe a map that takes the names of the seven dwarfs to the letters of the alphabet, by the rule that each name gets sent to its first letter.

- A map is called *injective* (or *one-to-one*) if different items in the input always get sent to different items in the output.

  Which of these maps is injective?

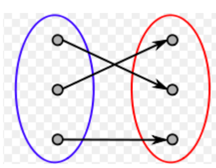
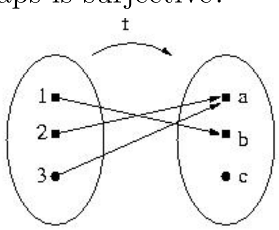



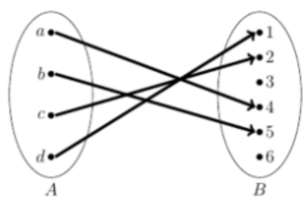


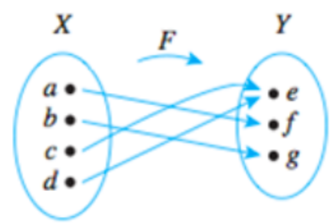


- A map is called *surjective* (or *onto*) if every item in the output gets hit by some item in the input.

  Which of these maps is surjective?

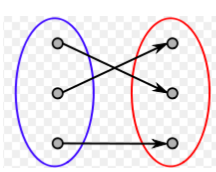
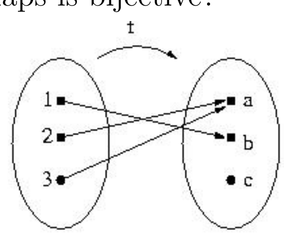



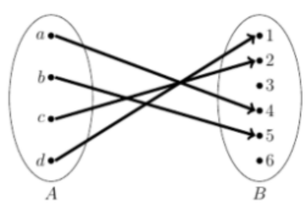


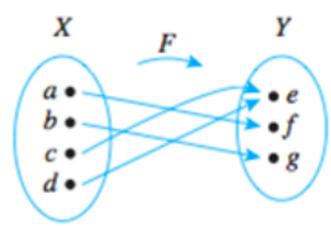


- A map is called *bijective* if it is both injective and surjective.

  Which of these maps is bijective?





  Is the first letter map from dwarf names to letters bijective?

4. Is the "plus 1" map from positive integers to positive integers injective? surjective? bijective?

5. What about the plus 1 map from the set of all integers ("Z") to the set all integers?

Is the first letter map from dwarf names to letters injective? surjective? bijective?

6. Which of these maps are injective? surjective? bijective?

   (a) The map from the 50 states to their capitals.

   (b) The map from all the people in this room to their birth months.

   (c) The map from the integers to the integers by $x \to 2x$.

   (d) The map $x \to x^2$ from the set of real numbers to the set of real numbers, by taking a number to its square, for example 3 to 9, etc.

3 Cardinality

7. We call two sets the same size, or *cardinaltiy*, if it is possible to find a bijective map (also called a *bijection*) between them. There might also be lots of maps between them that aren't bijections, but if there is some way to make a bijection between them, then they are the same size.

8. Which pairs of sets are the same cardinality?

	A	B
1)	positive odd integers	positive even integers
2)	positive integers	positive even integers
3)	positive integers and 0	positive integers
4)	positive integers	all integers
5)	positive integers	positive fractions
6)	positive fractions	positive real numbers

4 Facts

- The *c*ardinality of a set can be thought of as the size or number of elements in the set.
- The cardinality of the set of positive integers is denoted $\aleph_0$. Sets with cardinality $\aleph_0$ are called *countable* sets.
- Two sets have the same cardinality if there is a bijection from one set to the other. A *bijection* is a correspondence from elements of A to elements of B that is both *injective* (one-to-one) and *surjective* (onto).
- If there exists an *injection* from A to B but there does not exist any *bijection* from A to B, the we say that A has (circle one) smaller / bigger cardinality than B.
- If there exists an *surjection* from A to B but there does not exist any *bijection* from A to B, the we say that A has (circle one) smaller / bigger cardinality than B.
- The Schroder-Bernstein Theorem says that if there exists an *injection* from set A to set B and there also exists an *injection* from set B to set A, then A and B have the same cardinality.

5 Bijection Problems

9. Find a bijection between the set of positive integers and the set of positive integers that are divisible by 7.

10. Find a bijection between the set of positive integers and the set of positive integers that end in the digit 1.

11. Find a bijection between the set of positive integers that are divisible by 7 and the set of positive integers that end in the digit 1.

12. Find a bijection between the set of real numbers between 0 and 1 and the set of real numbers between 0 and 2.

13. Find a bijection between the set of real numbers between 0 and 1 and the set of real numbers between -1 and 1.

14. Find a bijection between the set of real numbers between 0 and 1 and the set of all positive real numbers.

15. Find a bijection between the set of all real numbers and the set of positive real numbers.

16. Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^2$ be the set of ordered pairs (x, y) where x is a real number and y is a real number. Does $\mathbb{R}^2$ have the same cardinality as $\mathbb{R}$?

17. The *power set* of a set S is written $\mathbb{P}(S)$ and means the set of all subsets of the set. For example $\mathbb{P}(\mathbb{R})$ contains $\{1, 2, 3, \ldots\}$, and $\{\pi, e, 1/2\}$ and many many other sets of real numbers. Is the power set of the real numbers the same size as the set of real numbers, or bigger? *Hint: suppose that there is a*

bijection f from $\mathbb{R}$ to $\mathbb{P}(\mathbb{R})$. Call a real number x "sticky" if $f(x)$ (which is a subset of real numbers) contains x. Now consider the subset of real numbers N that contains all non-sticky numbers. Since f is a bijection, there must be a number y for which $f(y)$ is that subset N. Is y sticky?

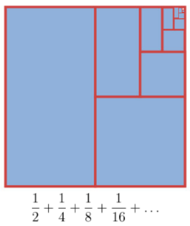
18. What is the cardinality of the set of infinite sequences of positive integers? For example, $1, 2, 15, 97, 46, 2, 3, 15, 9782, 141117, 4, 2, 1, \ldots$ would be one infinite sequence, and $5, 6, 8, 19, 37, 4999, 2, 5, 8598, \ldots$ would be another.

19. What is the cardinality of the set of infinite sequences of real numbers?

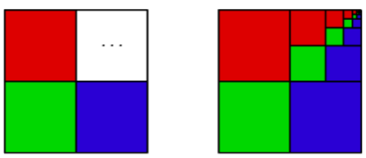
Reference: *Infinity and the Mind*, by Rudy Rucker

6 Picture Proofs of Geometric Series

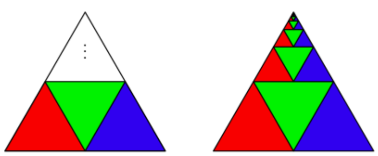
20. How does this picture help you find the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ (where the $\cdots$ means you keep adding up numbers in this pattern forever?



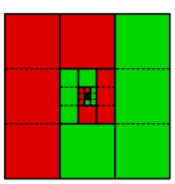
21. What infinite sum can you add up with this picture, and what does it sum to?



22. What infinite sum can you add up with this picture, and what does it sum to?



23. What infinite sum can you add up with this picture, and what does it sum to?



24. What infinite sum can you add up with this picture, and what does it sum to?