CHMC Advanced Group: Matroids

10/26/2019

1 Introduction

Matroids are a generalization of the notion of independence. They encode certain structure about sets of objects that relate to one another, and are found in a variety of areas of mathematics, spanning from combinatorics to geoemtry. Objects that have many formulations across different specialties of math are of great interest to many, as they provide a useful set of tools for linking different concepts and performing a variety of different types of analysis in different contexts. This worksheet will explore a few different formulations, though far from exhaustive, and explore some of their properties through different examples.

2 Matroids

The notion of a matroid can be given in several equivalent ways. Here we will present two such formulations. Before, we will motivate the notion of a matroid.

For a vector space V (think of $\mathbb{R}^2$ the plane or $\mathbb{R}^3$ three-space) and a field F (think $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) and non-zero vectors $v_1, \ldots, v_k \in V$, we say these vectors are linearly dependent if there exist coefficients $a_1, \ldots, a_k \in F$ not all zero such that $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$, where here 0 means the zero vector. If no such collection of coefficients exist, that is, if $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$ happens only when all the coefficients equal zero, then we say the collection is linearly independent.

Consider the following matrix, with real coefficients.

$$A = \begin{pmatrix} 1 & 3 & 5 & 6 \\ 2 & 6 & 6 & 8 \\ 1 & 3 & 2 & 3 \end{pmatrix}$$

Denote the columns by 1,2,3,4. Let E be the set of column labels $E = \{1, 2, 3, 4\}$ and denote by $\mathcal{I}$ the sets of column indices corresponding to sets of columns in the A that are linearly dependent. Note that $\mathcal{I}$ is not empty. Also, any subset of a set of linearly independent columns is again linearly independent. Finally, for any sets of linearly independent columns X and Y such that $|X| = |Y| + 1$, there is an index in $i \in X \backslash Y$ $Y \cup \{x\}$ is also a linearly independent set of columns.

Exercise 2.1 In the matrix above, determine the possible sets of linearly independent columns and verify the properties above.

Let E be a non-empty, finite set of elements. Denote by $\mathcal{I}$ a collection of subsets of E called independent sets that satisfy the following properties:

i) $\mathcal{I}$ is non-empty.

ii) Every subset of a set in $\mathcal{I}$ is also in $\mathcal{I}$.

iii) If X and Y are in I and $|X| = |Y| + 1$, then there is an element $x \in X \backslash Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.

Then, we say $M = (E, \mathcal{I})$ satisfying the above properties is a matroid.

From the example with the columns from the matrix M above, we see that there exists a matroid structure. The following example again illustrates a matroid.

Let $E = \{1, 2\}$ and set $\mathcal{I} = \{\emptyset, \{1\}, \{2\}\}$. We observe that $\mathcal{I} \neq \emptyset$, so axiom one is satisfied. The subsets of $\{1\}$ are $\{1\}$ and $\emptyset$, and similarly for $\{2\}$ are $\{2\}$ and $\emptyset$. Moreover, the only subset of $\emptyset$ is $\emptyset$. We see all of these subsets are included in $\mathcal{I}$ so axiom two is satisfied.

Finally, $|\emptyset| = 0$ and $|\{1\}| = |\{2\}| = 1$. Verifying that $1 \in \{1\} \backslash \emptyset$ and $\{1\} \cup \emptyset = \{1\} \in \mathcal{I}$ and similarly for $\emptyset$ and $\{2\}$, this satisfies the axiom three. Hence, $M = (E, \mathcal{I})$ determines a matroid.

Exercise 2.2 Let $E = \{1, 2, 3, 4\}$. Suppose that $\{1, 2\}$ and $\{3, 4\}$ are known to be in $\mathcal{I}$ and that $\mathcal{I}$ contains no subsets of E that are of cardinality three or greater. Determine what other sets must be in the collection of independent sets $\mathcal{I}$ be to make $E, \mathcal{I}$ a matroid? Use the above axioms to determine the sets of size 0, 1, and 2 that must be in $\mathcal{I}$. Note, there may be several different ways to extend this to a matroid, although there are some subsets that will be common to all.

Exercise 2.3 From the previous exercise, examine all the independent sets that are maximal (not contained in any other indpendent set). What do you notice? Try to conjecture a statement about the size of the maximal independent sets and prove it.

An alternative formulation for a matroid is the following.

Let E be a non-empty finite set. A collection of bases, $\mathcal{B}$ is a collection of subsets of E. A matroid M is a pair $M = (E, \mathcal{B})$ where $\mathcal{B}$ is a collection of bases satisfying the following properties:

i) No $X \in \mathcal{B}$ properly contains any $Y \in \mathcal{B}$.

ii) If B_1 and B_2 are bases and if $e \in B_1$ there is an $f \in B_2$ (different from e) such that $B_1 \backslash \{e\} \cup \{f\}$ is also a base.

The following exercise will relate this notion of a matroid to the previous notion, by defining one of the possible matroids (the largest) from exercise 2.2.

Exercise 2.4 Verify that for $E = \{1, 2, 3, 4\}$, that $\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ is a collection of bases satisfying the required properties.

Exercise 2.5 From the previous exercise, can you notice a property about the sizes of the bases defining the matroid? Try to prove your claim. For this, property ii is very useful.

Exercise 2.6 Try to relate the independent sets from exercise 2.2 with the bases from 2.4. What do you observe? Try to create a connection between the two different formulations of matroids bases on the properties of independent sets and bases, and the individual independent sets and bases themselves.

A bijection of sets A and B is a function $f : A \to B$ such that:

i) If $b \in B$ then there is an $a \in A$ such that $f(a) = b$;

ii) For $x, y \in A$ if $f(x) = f(y)$ then $x = y$.

Note that if a bijection exists, then there is an inverse function $g : B \to A$, defined by $g(b) = a$ iff $f(a) = b$, that has the same properties as f, with the roles of A and B switched.

We say two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are isomorphic if there is a bijection between E_1 and E_2 such that independent sets are mapped to independent sets by f and by g.

Exercise 2.7 For $E = \{1, 2\}$ and matroids M_1 defined by $\mathcal{I}_1 = \{\emptyset\}$, M_2 defined by $\mathcal{I}_2 = \{\emptyset, \{1\}\}$, M_3 defined by $\mathcal{I}_3 = \{\emptyset, \{2\}\}$, M_4 defined by $\{\emptyset, \{1\}, \{2\}\}$ and M_5 defined by $\mathcal{I}_5 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}$, which matroids are isomorphic?

Exercise 2.8 Determine how many non-isomorphic matroids on the set $E = \{1, 2, 3\}$ exist. To do this, first determine which collections $\mathcal{I}$ of independent sets form matroids with the set E. Then, determine which ones are isomorphic.

Recall the example of the matroid determined by the sets of linearly independent columns of the matrix A, which we will denote by $M[A]$. A matroid that is isomorphic to such a matroid for a given matrix A over a field F is said to be $F-$representable, and A is an $F-$representation. It is a fact that not all matroids are $F-$representable over every field F.

3 Graphs and matroids

A **simple graph** $G = (V, E)$ is a set V of distinct vertices and a set E of distinct edges with certain restrictions. More precisely, we will look at **finite graphs** for which V and E are finite. Associated to each edge $e \in E$ is an unordered pair of distinct vertices $\{v_i, v_j\}$; we do not care about the direction the edge travels, just the vertices it connects. We also will only consider graphs where any pair of vertices is connected by at most one edge. A **path** from v_i to $v_j \neq v_i$ in G is given by a sequence of edges such that each pair of adjacent edges shares a common vertex, with the first edge having v_i as a vertex and the last edge having v_j as a vertex. A graph G is **connected** if for every pair of vertices, there exists a path between them. A **cycle** is a non-trivial path that starts and ends at the same vertex.

A **tree** is a connected graph that contains no cycles. If a tree has n vertices, one can show that it has $n - 1$ edges. A **spanning tree** for a graph G is a tree that uses the edges of G and includes every vertex of G. Every connected graph has a spanning tree, but there may be many such spanning trees for a given graph.

Consider the complete graph on n vertices. This a graph with n vertices and an edge in between each vertex pair.

Exercise 3.1 List out the spanning trees for $n = 3$. How many are there? How about for $n = 4$?

A **forest** is graph that contains no cycles. A tree is a connected forest.

Exercise 3.2 Determine the possible forests in the complete graph on 3 vertices.

We can define a matroid on a graph G by letting E be the set of edges and the set of independent sets $\mathcal{I}$ be the edge sets of forests in the graph G. Such a matroid is said to be a **cycle matroid**, denoted by $M(G)$.

Exercise 3.3 Verify that the collection of edge sets of forests on the complete graph on 3 vertices satisfies the properties of a collection of independent sets.

A matroid M is a **graphic** if it is isomorphic to a cycle matroid for some graph.

We may define a matroid in a third formulation as follows. Let E be a finite non-empty set and C and non-empty collection of subsets of E satisfying the following two properties:

i) No cycle properly contains another cycle.

ii) For two cycles C_1 and C_2 such that $e \in C_1 \cap C_2$, there is a cycle $C_1 \cup C_2$ not containing e.

Exercise 3.4 For the complete graph on three vertices, list out the cycles. Do the same for the complete graph on four vertices.

Exercise 3.5 Examine the formulation of a matroid using cycles and look at the collection of cycles you created from the previous exercise. What similarities do you see? Also, examine the formulation of a matroid in terms of independent sets. How does this formulation relate the independent sets to the cycles you found in the previous exercise?

4 Operations with matroids

Now that we have seen several different formulations of matroids and examples of these, one may ask if there are ways of building new matroids from existing matroids.

We introduce the direct sum of two matroids as follows:

Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be two matroids given by collections of independent setsm where E_1 and E_2 are disjoint ($E_1 \cap E_2 = \emptyset$). Then

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 : \; I_1 \in \mathcal{I}_1, \; I_2 \in \mathcal{I}_2\})$$

Exercise 4.1 Verify that the three properties that independent sets must satisfy to form a matroid are in fact satisfied in this construction. Hint: If $|I_1 \cup I_2| = |J_1 \cup J_1| + 1$, it must follow that either $|I_1| > |J_1|$ or that $|I_2| > |J_2|$. Use this and property 2 to verify property 3.

Observe that if G_1 and G_2 are disjoint graphs, then the cycle matroids determined by these graphs can be combined using the direct sum process.

Exercise 4.2 Let G_1 and G_2 be two disjoint complete graphs on 3 vertices, and $M(G_1)$ and $M(G_2)$ be the corresponding cyclic matroids. Write out the independent sets of $M(G_1) \oplus M(G_2)$. It may be helpful to consider labelling the vertices and edges of G_1 as v_i, e_i and the vertices and edges of G_2 as w_i, f_i.

The direct sum of two $F-$representations is also defined. To see this, suppose that $M_1 = M[A_1]$ and $M_2 = M[A_2]$. Then, consider the matrix

$$A_3 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

Exercise 4.3 Let

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

Determine the matroids $M[A_1]$ and $M[A_2]$. Next, determine $M[A_1] \oplus M[A_2]$ using the definition of the direct sum of matroids. Then, verify that this is isomoprhic to $M[A_3]$, where

$$A_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \end{pmatrix}.$$

It may be easier to keep track of everything by labelling the columns of A_1 by $\{1, 2, 3\}$, the columns of A_2 by $\{4, 5, 6\}$, and the columns of A_3 by $\{1, 2, 3, 4, 5, 6\}$.

Solution: $M[A_1]$ is given by $E_1 = \{1, 2, 3\}$ and $\mathcal{I}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. $M[A_2]$ is given by $E_2 = \{4, 5, 6\}$ and $\mathcal{I}_2 = \{\emptyset, \{4\}, \{5\}, \{6\}, \{4, 6\}, \{5, 6\}\}$. One can easily compute the direct sum from this and verify it equals $M[A_3]$, though this is tedious (there are 42 independent sets).

References

1. https://www.math.lsu.edu/ oxley/survey4.pdf

2. https://www.whitman.edu/Documents/Academics/Mathematics/hillman.pdf