CHMC Advanced Group:
Group Actions and Symmetry

08/11/2018

1 Introduction

In mathematics, to study certain objects one must employ a variety of techniques. For a discrete object, one may consider the symmetries of that object and these in turn will determine a group of symmetries. However, one may also ask how does a specified group act on the object. In trying to determine what action may exist and what the action looks like, properties of the object in question come to light.

Of particular interest in this worksheet is the action of a specific group, the symmetric group on n letters S_n, on polynomials in n variables, $p(x_1, \ldots, x_n)$. The polynomials fixed by this action are the symmetric polynomials and have very interesting properties. In addition, they appear in several areas of mathematics so appear to encode some fundamental aspects of the underlying structure of those particular contexts. We will explore this throughout the worksheet.

2 Symmetric group

Suppose you have n balls, labelled $1, 2, \ldots, n$ and n distinguished positions. Placing the balls such that one and only one ball is in each distinguished position, describes a permutation of the balls. In particular, for each number represented on the ball, we have specified a number, the position of that ball. Consider, for example, three balls B_1, B_2, B_3 placed into positions P_1, P_2, P_3 such that B_1 is in P_2, B_2 is in P_1 and B_3 is in P_3. This has specified that 1 goes to 2, 2 goes to 1 and 3 goes to 3. One way to represent this is in the following manner: (1 2)(3). To read this, start at left of the first parenthesis and read left to right. (1 2) encodes that 1 goes to 2. Since 2 is the rightmost element of the parentheses, read this as 2 is sent to the first element in the parentheses, namely 1. Next, (3) indicates that 3 is sent to itself.

Consider the following example: There are four balls $B_1, \ldots, B_4$ and four positions $P_1, \ldots, P_4$ where B_1 is in P_3, B_2 is in P_1, B_3 is in P_2, and B_4 is in P_4. This means 1 is sent to 3, 3 is sent to 2, 2 is sent to 1, and 4 is sent to 4. Thus, we encode this as (132)(4).

Exercise 2.1 Consider the case where there are five balls and five positions. Suppose B_1 is in P_2, B_2 is in P_4, B_3 is in P_5, B_4 is in P_1, and B_5 is in P_3. What is the representation of this using parentheses?

Exercise 2.2 Suppose you are given a permutation representing a placement of balls in positions, say (1 2)(3 5 4). Where is each ball B_i placed in terms of positions P_i? What about the representation (1 2)(5 4 3) or (2 1)(4 3 5)? How about (3 5 4)(1 2)? What do you observe is happening with all of these representations?

Now consider giving a permutation of the positions of the balls and following that with another permutation of the positions. For example, suppose we set $\sigma = (1\ 2)(3)$ in the first example and $\tau = (1\ 2\ 3)$. The permutation σ sends B_1 to P_2, B_2 to P_1 and B_3 to P_3 while τ sends B_1 to P_2, B_2 to P_3, and B_3 to P_1. If we first specify the positions of the balls by σ and then specify the position of the balls using τ and the positions specified by σ, the following is happening: σ sends B_1 to P_2 and τ send this to P_3, σ sends B_2 to P_1 and τ sends this to P_2, and σ sends B_3 to P_3 with τ sending this to P_1. Hence, $\tau\sigma$ sends B_1 to P_3, B_2 to P_2 and B_3 to P_1, encoded by (1 3)(2). However, we could have also considered $\tau\sigma = (1\ 2\ 3)(1\ 2)(3)$ and similarly arrived at (1 3)(2). To do this, we track where each number is sent, by reading right to left. Start with 1. The first 1 encountered is in (1 2), so 1 is sent to 2, and we continue to the next parentheses where we see 2 is sent to 3. There are no more 3*s* so we conclude 1 is sent to 3. Now, look at 3. (3) tells us that 3 is sent to 3 and (1 2 3) tells us 3 is sent to 1, with nothing else after. Hence 3 is sent to 1 and so we close the parentheses since we have returned back to 1, yielding (1 3). Now, look at 2. (1 2) tells us it is sent to 1 and (1 2 3) tells us it is sent back to 2. Hence, we get (2). Thus, in total we have (1 3)(2). From the above exercise, since the elements of the groupings are all distinct, we could also have written this as (2)(1 3), (2)(3 1) or (3 1)(2).

Exercise 2.3 If $\sigma = (1\ 2\ 4)(3\ 5)$ and $\tau = (1\ 2)(3\ 4)(5)$, calculate $\sigma\tau$ and $\tau\sigma$. Is it true that $\sigma\tau = \tau\sigma$? Do the same for $\sigma = (1\ 2\ 3\ 4\ 5)$ and $\tau = (1\ 3\ 5\ 2\ 4)$. In this case, also compute $\sigma\sigma$. What does this look like?

We sometimes use the notation $\sigma(i) = j$ to denote that the permutation σ sends i to j. For example, if $\sigma = (1\ 2\ 3)$, then $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. In addition, for distinct integers $a_1, \ldots, a_k$ in the set $\{1, 2, \ldots, n\}$, the permutation denoted by $(a_1\ a_2, \cdots a_k)$ is called a cycle of length k. A cycle of length 2 is called a transposition. Notice that (1 2)(1 3) sends 1 to 3, 2 to 1, and 3 to 2, so can be rewritten in the form (1 3 2). In the next exercise, you will show that this holds for any cycle.

Exercise 2.4 Show that every cycle of length $2 \leq k \leq n$ can be constructed as the product of transpositions. Hint: Assume this holds for all cycles of length $k - 1$ where $2 \leq k \leq n$. Then, for a given cycle of length k, try constructing it as a product of a transposition and a cycle of length $k - 1$.

Define S_n to be the set of permutations on n letters. Every such rearrangement of the balls $B_1, \ldots, B_n$ into positions $P_1, \ldots, P_n$ defines a permutation on n letters and is thus in S_n. In the next section, we introduce some additional structure to S_n, which allows us to call it a group.

3 Group actions

We now discuss group actions.

Definition. *A group G is a set with an operation $\cdot$ such that:*

i). For any two elements $a, b \in G$, $a \cdot b$ is another element in G.

ii). For any $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

iii). There is a unique element $e \in G$ such that for any $g \in G$, $e \cdot g = g = g \cdot e$.

iv). For each $g \in G$, there is an element $g^{-1} \in G$ such that $g \cdot g^{-1} = e = g^{-1} \cdot g$.

In the above definition, property ii actually tells us that the inverse g^{-1} to g given in property iv is actually unique.

Exercise 3.1 For any $g \in G$, show that there is a unique inverse g^{-1} satisfying $g \cdot g^{-1} = e = g^{-1} \cdot g$. Hint: Suppose $h, k \in G$ also satisfied this and then show $h = k$.

Recall in the previous section, we saw that $\sigma, \tau \in S_n$ were group elements of permutations where $\sigma\tau$ was also a permutation, hence $\sigma\tau \in G$.

Exercise 3.2 Let $\sigma \in S_3$ be given by $\sigma = (12)(3)$. What is σ^2? In general, if $\tau = (ab)$ for $a \neq b$, what is τ^2?

Suppose $\sigma \in S_4$ is given by $\sigma = (1234)$. What is σ^{-1}? Consider that $e = (1)(2)(3)(4) = \sigma\sigma^{-1}$, so $\sigma\sigma^{-1}(1) = 1$, $\sigma\sigma^{-1}(2) = 2$ etc. Where must σ^{-1} send 1, 2, 3, 4?

We now consider how a group "acts" on objects. In considering the distinguished balls $B_1, \ldots, B_n$ and distinguished positions $P_1, \ldots, P_n$, we encoded a permutation of the balls into various positions by an element of the form $(a_1, a_2, \ldots, a_{k_1}) \cdots (a_{k_l+1}, \ldots a_n)$. However, we could have instead considered that the permutation acted on the balls B_1 in position P_1, $\ldots$, B_n in position P_n by sending B_1 to position $\sigma(1)$, B_2 to position $\sigma(2)$, and so on. In this sense, the group element σ is acting on the positions of the balls B_i.

There are many such examples of group actions and we will focus on how the symmetric group acts S_n acts on polynomials in n variables. Before doing so, it is important to define what a group action is more precisely than we have seen thus far.

Definition. *Given a group G, a left group action on X is a way of associating to a pair (g, x) of a group element $g \in G$ and object $x \in X$ an element x' subject to the following axioms:*

i). $e \cdot x = x$ for all $x \in X$.

ii). $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $x \in X$ and $g, h \in G$.

Now, consider the action of the symmetric group S_n of a polynomial of n variables, $x_1, \ldots, x_n$. Define $\sigma \cdot p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$.

For example, let $p(x_1, x_2) = x_1^2 + x_2$ and let $\sigma = (12)$. Then $\sigma \cdot p(x_1, x_2) = p(x_{\sigma(1)}, x_{\sigma(2)}) = p(x_2, x_1) = (x_2)^2 + x_1$. For another example, suppose $p(x_1, x_2, x_3) = x_1 - x_2 + x_3$. Then, $(12) \cdot p(x_1, x_2, x_3) = x_2 - x_1 + x_3$ while $(13) \cdot p(x_1, x_2, x_3) = x_3 - x_2 + x_1 = x_1 - x_2 + x_3$.

Exercise 3.3 Let $p(x_1, x_2, x_3) = x_1 x_2 + x_3^2$. Determine the action of S_3 on $p(x_1, x_2, x_3)$. What polynomials do you get? Which permutations change the polynomial $p(x_1, x_2, x_3)$? Which ones leave the polynomial unchanged?

Exercise 3.4 Now, consider the polynomial $p(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$. Look at the action of S_3 on this. What do you observe?

Exercise 3.5 Do the same analysis on the polynomial $p(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. What do you observe? How do permutations of the form $(a_1 a_2)(a_3)$ act on the polynomial? How about permutations of the form $(a_1 a_2 a_3)$?

We introduce the concept of the sign of a permutation. Recall that in the previous section you showed that every cycle can be written as a product of transpositions. We define $\text{sgn}(\sigma) = (-1)^k$, where σ is a permutation that can be written as the product of k transpositions. This is well-defined since the parity (evenness or oddness) of the number of transpositions whose product is a given permutation is always the same. For a permutation which is the product of several cycles consisting of disjoint elements $\sigma = (a_1\ a_2\ \cdots a_k)(b_1 \cdots b_l) \cdots (h_1 \cdots h_j)$, define the $\text{sgn}(\sigma) = \text{sgn}((a_1\ \cdots a_k)) \cdots \text{sgn}((h_1 \cdots h_j))$.

Exercise 3.6 What is the sign of a cycle of length 2, 3, 4? In the previous exercise, how does the sign of the permutation relate to how the permutation acted on the polynomial $p(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$?

Sometimes, instead of writing a polynomial in its factorization

$$p(x) = a_n (x - r_1)(x - r_2) \cdots (x - r_n),$$

we write

$$p(x) = a_n \prod_{i=1}^{n} (x - r_i)$$

where a_n is the leading coefficient of the polynomial and the r_i are the roots of the polynomial.

Exercise 3.7 For a polynomial in n variables $p(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3) \cdots (x_{n-1} - x_n)$, what is the effect of acting by a transposition $\tau = (a_1\ a_2)$ on the polynomial? Conclude that one can recover the sign of a permutation σ by looking at $\sigma \cdot p(x_1, \ldots, x_n)$ in relation to $p(x_1, \ldots, x_n)$.

You may have noticed that in the previous exercises, there were polynomials that did not change under the action of S_n and there were polynomials that changed under S_n in particular ways. We explore these actions more in the next section.

4 Symmetric Polynomials

Let $p(x_1, \ldots, x_n)$ be a polynomial in n variables as in the previous section and S_n the symmetric group on n letters. We begin with a definition.

Definition. *A polynomial $p(x_1, \cdots, x_n)$ in n variables is symmetric if $\sigma \cdot p(x_1, \cdots, x_n) = p(x_1, \ldots, x_n)$ for every $\sigma \in S_n$.*

In the terminology of group actions, the symmetric polynomials are the fixed points of the action of S_n on the polynomials of n variables.

Exercise 4.1 Consider the case of $n = 3$ and the polynomial $p(x_1, x_2, x_3) = x_1 - x_2^2 + x_3$. Is this polynomial symmetric? If not, what are all the polynomials that arise from the action of S_3 on p?

Exercise 4.2 Suppose we want to construct a symmetric polynomial from some term. In particular, suppose we want a polynomial $p(x_1, x_2, x_3)$ that has a term x_1x_3 in it. What other terms must necessarily be included in this polynomial? Consider the action of S_3 on the term x_1x_3. Add up all the distinct terms you get from this action. Is the resulting sum symmetric?

There are different sets of symmetric polynomials which are useful in the study of symmetric polynomials in general. One such set, the elementary symmetric polynomials, are important for a fundamental result of symmetric polynomials.

Definition. *The elementary symmetric polynomials in n variables are written $e_k(x_1, \ldots, x_n)$ for $k = 0, 1, \ldots, n$ and defined as*

$$e_0(x_1, \ldots, x_n) = 1$$

$$e_1(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

$$e_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots x_{n-1}x_n$$

and in general,

$$e_k(x_1, \ldots, x_n) = \prod_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k}.$$

For $k > n$, $e_k(x_1, \ldots, x_n) = 0$.

In the above definition, the symbol

$$\sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k}$$

means consider the sum of all $k-$term products of distinct variables x_i.

To illustrate this definition, $e_1(x_1, x_2, x_3) = x_1+x_2+x_3$, $e_2(x_1, x_2, x_3) = x_1x_2+x_1x_3+x_2x_3$, and $e_3(x_1, x_2, x_3) = x_1x_2x_3$.

Exercise 4.3 Assume $n \geq 1$ and $0 \leq k \leq n$. How many terms are there in the expression for $e_k(x_1, x_2, \ldots, x_n)$? In the above example for $e_k(x_1, x_2, x_3)$, there are 3 terms for $k = 1, 2$ and 1 term for $k = 0, 3$. Try calculating the number of terms for $e_k(x_1, x_2, x_3, x_4)$ and see if you can relate the number of terms to other mathematical concepts.

Exercise 4.4 Suppose a polynomial $p(x) = \prod_{i=1}^{n}(x - r_i)$ is written in the form $p(x) = x^n+a_{n-1}x^{n-1}+\cdots a_1x+a_0$. Can you relate the value of each coefficient a_i with the elementary symmetric polynomials e_k evaluated on $r_1, \ldots, r_n$? In other words, can you write a_i in terms of $e_1(r_1, \ldots, r_n), e_2(r_1, \ldots, r_n), \ldots, e_n(r_1, \ldots, r_n)$? Try this for the cases when $n = 2, 3, 4$ and generalize.

There is an important theorem about symmetric polynomials that I present without proof, called The Fundamental Theorem of Symmetric Polynomials.

Theorem Every symmetric polynomial $P(x_1, \ldots, x_n)$ can be written in the form

$$Q(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n))$$

where $Q(y_1, \ldots, y_n)$ is a polynomial in n variables.

To illustrate this, consider the polynomial $p(x_1, x_2) = x_1^2 + x_2^2$. We can write this as $p(x_1, x_2) = (x_1 + x_2)^2 - 2(x_1x_2) = e_1(x_1, x_2)^2 - 2e_2(x_1, x_2)$. In this case, $Q(y_1, y_2) = y_1^2 - 2y_2$.

Exercise 4.5 Using the above exercise, if r_1, r_2, r_3 are the three roots of $x^3 + 3x^2 - 2x + 1$, determine the value of $r_1^2 + r_2^2 + r_3^2$. Hint: Since $r_1^2 + r_2^2 + r_3^2$ is a symmetric polynomial in three variables, it can be written as a polynomial in the elementary symmetric polynomials. Use what you know from the previous exercise about the value of the elementary symmetric polynomials on the roots of a polynomial to determine the value of $r_1^2 + r_2^2 + r_3^2$.

Exercise 4.6 Continuing with the previous exercise, determine the value of $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. It may be useful to find a common denominator and determine the values of the numerator and denominator.

5 Applications

We present an application.

Suppose to an object V, there is a polynomial associated of the form

$$c_t(V) = \prod_{i=1}^{n}(1 + r_i t).$$

We define $c_i(V)$ to be such that

$$c_t(V) = 1 + c_1(V)t + c_2(V)t^2 + \cdots + c_n(V)t^n.$$

In addition, we also associate to V a second polynomial in the values of $r_1, \ldots, r_n$,

$$ch(V) = (e^{r_1} + \cdots + e^{r_n})_n = \sum_{i=0}^{n} \frac{r_1^i + \cdots + r_n^i}{i!}.$$

where we truncate the infinite series e^s to terms of powers less than or equal to n. This can be more compactly written as

$$ch(V) = ch_0(V) + ch_1(V) + \cdots + ch_n(V)$$

where

$$ch_i(V) = \frac{r_1^i + \cdots + r_n^i}{i!}.$$

These polynomials are called the Chern Polynomial and the Chern Character associated to V and appear in the areas of Topology and Algebraic Geometry among many areas of mathematics. We call $c_i(V)$ are the i^{th} Chern class of V and $ch_i(V)$ the i^{th} Chern character of V.

Exercise 5.1 Write the Chern classes of V, $c_i(V)$ in terms of the values of $r_1, \ldots, r_n$. After that, try expressing the values of $ch_1(V)$, $ch_2(V)$, and $ch_3(V)$ in terms of the values of the Chern classes of V. Also, what is the value of $ch_0(V)$? Consider continuing your calculation of the $ch_i(V)$ for larger i. The calculations become more cumbersome the larger values of i.