CHMC Advanced Group:
Pigeonhole Principle

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1 Introduction

Combinatorics is an area of math focused on counting specific objects, sometimes in multiple ways. While the theory represented by combinatorics can become complicated very quickly and covers a wide variety of theoretical subfields, some basic tools of combinatorics are quite powerful in simplifying complex problems. One of these tools is the Pigeonhole Principle. In this worksheet, we will first learn about what pigeonhole principle is in some simple contexts, some more complex contexts, and finally look at some more complicated applications of it.

2 Basics on Pigeonhole Principle

Consider the following scenario. You have a dresser drawer full of socks, say black socks and white socks. You always wear a matching pair of socks, either both black or both white. If you reach into your drawer without looking, and remove one sock at a time, how many socks do you need to pull out to guarantee that you have a matching pair? Well, suppose you pull out two consecutive black socks or two consecutive white sock. One might consider two socks to be the answer. However, if instead you pull out a black sock and then a white sock, you have removed two socks with no matching pair. However, the next sock however will create a matching pair with either a pair of black socks or a pair of white socks.

Exercise 2.1 Suppose instead that you have a drawer full of socks of $m \geq 2$ different colors. If you want to wear a pair socks of the same color, what is the least number of socks you need to remove from the drawer to guarantee this?

The idea behind this sort of counting problem is call the Pigeonhole Principle. The basic form of this principle states that if n objects are placed into m boxes, with $n > m > 0$ integers, then at least one of the m boxes contains at least 2 objects.

In the previous exercise, we consider m boxes corresponding to each sock color. A matching pair is realized when we have two socks of the same color. By pigeonhole principle, for $n > m$ choices of socks, there will be at least one box with at least two socks. Hence, the least such n is $m + 1$, which you determined.

The application of Pigeonhole Principle is not too hard; determining how to label the objects and boxes is the trickiest part.

Exercise 2.2 A mathematics class is given an exam, with possible scores from 0 to 100. How many people must be in the class to guarantee that at least two people share the same score on the exam? For this, what should the objects be? How about the boxes?

Now for another application of Pigeonhole Principle.

Exercise 2.3 Suppose there is a street with 10 houses on one side, with address $2, 4, 6, \ldots, 20$. On Saturday, a truck delivering packages stops at different houses among the 10 houses. How many houses must the truck stop at to guarantee that a consecutive pair of houses receive a package? What should the objects be? How about the boxes? Consider drawing out a few examples if you get stuck.

One thing to keep in mind when approaching these problems is how "bad" distributing the objects among the boxes can be. In the second exercise, we might consider the "worst" possible outcome to be pulling out a different sock for the first m socks. In this case, there is a sock of each color represented with no pairs created. Once we hit this point, any additional sock will automatically create a pair.

3 Advanced concepts of Pigeonhole Principle

Now consider the following scenario. Suppose instead of 2 feet, you have $k \geq 2$ feet and a sock full of black socks and white socks. You always wear socks that match, so either all black socks or all white socks.

Exercise 3.1 How many socks do you need to remove from your drawer of black and white socks to guarantee that you have k black socks or k white socks to wear? Consider what the "worst" possible scenario would be in this case.

The Generalized Pigeonhole Principal states that if there are n objects and m boxes, such that $n \geq km + 1$ for $n, m, k > 0$ integers, then at least one box will have at least $k + 1$ objects in it. Intuitively we can reason this out by considering placing k objects into each box. This requires km objects and is considered the "worst" possible scenario. However, the addition of one more object will force one of the boxes to contain $k + 1$ objects, so $km+1$ objects are required to guarantee that among m boxes, at least one has $k+1$ objects.

Exercise 3.2 Suppose there are 19 distinct points distributed in a square of side length 3 feet. Show that there are at least 3 points that fall within a square of side length 1 foot. Hint: Consider dividing the square of side length 3 feet into 9 squares of side length 1 foot.

In the next exercise, consider arguing by contradiction, that is assuming that the assertion is false and leading to a contradiction.

Exercise 3.3 Suppose that a group of fifteen squirrels collected a total of 100 acorns. Show that two squirrels must have collected the same number of acorns. Hint: Suppose each squirrel collected a distinct number of acorns. What is the smallest sum of acorns achieved in this way?

4 Additional Exercises

In this section we will see a variety of different types of exercises all of which have solutions that use the pigeonhole principle.

Exercise 4.1 15 chairs are placed around a circular table. On the table are name cards for 15 guests. After the guests sit down, it turns out that none of them is sitting in front of his own card. Prove that the table can be rotated so that at least 2 guests are simultaneously correctly seated.

Exercise 4.2 An ice cream shop serves 4 flavors of ice cream. 7 friends show up, and each of them orders a cone with 2 different flavors. Prove that there must be at least 2 people who ordered the same combination of flavors.

Note: combinations are unordered, unlike permutations. So, for example, if one person has a vanilla scoop on top of a chocolate one, that is equivalent to a chocolate scoop on top of a vanilla one. How would you change this exercise if you were instead considering permutations rather than combinations?

Exercise 4.3 If we choose 6 integers between 1 and 10, prove that at least two of them must be consecutive.

Related exercise, attributed to Erdős: Let n be a positive integer. Prove that any collection of $n+1$ distinct elements taken from the set $\{1, 2, \ldots, 2n-1, 2n\}$ must contain two elements which are relatively prime.

Exercise 4.4 Let S be any set of 20 distinct integers chosen from the arithmetic progression[1] $1, 4, 7, \ldots, 100$.

(a) Show that there are two number in S that sum to 101.

(b) Must there be two number in S that sum to 107? 110? 116? Give a proof of counterexample.

(c) For which N is it guaranteed that there will be two numbers in S that sum to N?

Exercise 4.5 Show that in a class of n students, there are at least two students with the same number of friends in the class. (Assume friendship is always mutual.)

Exercise 4.6 Let every point of the plane be painted white or red. Show that there is a rectangle whose vertices are all the same color.

Exercise 4.7 Let every point of the plane be painted white or red. Show that there are two points colored the same way and exactly 1 inch apart.

Exercise 4.8 Let every point of the plane be painted white, red, or blue. Show that there are two points colored the same way and exactly 1 inch apart.

Exercise 4.9 What is the minimal number of colors needed to color the plane so that no two points 1 inch apart are unicolor?

[1] Every member of an *arithmetic progression (or series)* is obtained from the previous member by adding the same number d. In our case, $d = 3$.

Exercise 4.10 Find a coloring of the plane with 7 colors such that any two points 1 inch apart are colored differently.

Exercise 4.11 Color the diagonals and sides in a (convex) hexagon red or blue. Show that the diagram contains an all-red or an all-blue triangle.

Exercise 4.12 Color the diagonals and sides of a (convex) pentagon in red and blue in such a way that no unicolor triangle is formed.

Exercise 4.13 Color the diagonals and sides in a convex 17-gon red, blue or green. Show that the diagram must contain a unicolor triangle.

Exercise 4.14 Let each diagonal and side of a convex n-gon be colored in one of k colors. What is the minimal number of vertices n to guarantee that the resulting diagram contains a unicolor triangle?

Exercise 4.15 Let n be a positive integer, and let L be an ordering of $\{1, 2, \ldots, n^2, n^2+1\}$. Prove that there must be a subsequence S in L of length at least $n+1$ such that S is strictly increasing or S is strictly decreasing.

Further, prove that this result is optimal in the following sense: for any positive integer n, there exists an ordering L of $\{1, 2, \ldots, n^2\}$ such that *no* subsequence of length $n+1$ is strictly monotonic.

Exercise 4.16 Let m, n be positive integers. Adapt the previous exercise to show that given any ordering of $\{1, 2, \ldots, mn, mn+1\}$ there is a subsequence S in L of length at least $n+1$ such that S is strictly increasing or a subsequence S' in L of length at least $m+1$ such that S' is strictly decreasing.

Exercise 4.17 There are a number of variants of the pigeonhole principle. For example, if you have infinitely many pigeons assigned to only finitely many holes, then at least one hole must have infinitely many pigeons. One can get even more general: if one has *uncountably* many pigeons assigned to *countably many* holes, then at least one hole must contain uncountably many pigeons.

(a) Let $(G, *)$ be a finite group. Prove that for any $g \in G$, g has finite order. That is, there exists some positive integer n such that $g^n = e$, the identity of the group.

(b) Let $A := \{a_i : i \in I\}$ be an uncountably infinite set of positive numbers. Prove that

$$\sum_{i \in I} a_i$$

cannot converge to a real number.

Hint: without loss of generality, we may assume A is bounded above by 1; that is, for all $i \in I$, $0 < a_i \leq 1$. Partition this interval as

$$\begin{aligned}(0,1] &= \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right] \\ &= \cdots \cup \left(\frac{1}{n+1}, \frac{1}{n} \right] \cup \cdots \cup \left(\frac{1}{4}, \frac{1}{3} \right] \cup \left(\frac{1}{3}, \frac{1}{2} \right] \cup \left(\frac{1}{2}, 1 \right].\end{aligned}$$

How might the pigeonhole principle apply to complete this proof?