Continued Fractions

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1 Puzzler

1. What number does this represent?

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

2 Notation

To save space, we will write the long expression

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

as the list

$$[1; 2, 2, 2, 2, 2, \cdots]$$

where the dots mean the pattern continues forever.

We can also write it as $[1; \overline{2}]$ where the bar on top means that the 2 repeats.

2. Write

$$3 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{7 \cdots}}}}$$

as a list.

3. Write out $[5; 1, 2, 3, 4, 5, 6, 7, \cdots]$ in long form form.

3 Definitions

A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where a_0 is an integer (that could be zero) and the other a_n's are positive integers (can't be zero).

The list $[a_0; a_1, a_2, a_3, \cdots]$ represents the same thing.

4. For the continued fraction

$$\cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \cfrac{1}{10 + \cdots}}}}}$$

what is a_0? a_1? a_2?

Write it as a list.

4 Evaluating Continued Fractions

5. Is $[0; 2, 4, 6, 8, 10 \cdots]$ bigger than 1 or less than 1? How could we evaluate it?

The fractions obtained by evaluating the initial pieces of a continued fraction are called the **convergents** of the partial fraction. For the example $[0; 2, 4, 6, 8, 10, \cdots]$, the first few convergents are:

$$0, 1/2, 4/9,$$

The terms (a_n), the convergents (c_n) and the numerators and denominators of the convergents (p_n and q_n) are listed in the table below. Fill in the rest of the table.

n	0	1	2	3	4	5
a_n	0	2	4	6	8	10
c_n	0	$\frac{1}{2}$	$\frac{4}{9}$			
p_n	0	1	4			
q_n	1	2	9			
decimal	0.00000000	0.50000000	0.44444444			

6. Work out the convergents for the continued fraction $[1; 1, 1, 1, 1, 1, 1, \cdots]$. What do you notice?

n	0	1	2	3	4	5
a_n						
c_n						
p_n						
q_n						
decimal						

7. Write down any continued fraction that you choose. Trade with your neighbor and evaluate your neighbor's continued fraction as a number (approximately). You can use this chart to help you.

n	0	1	2	3	4	5
a_n						
c_n						
p_n						
q_n						
decimal						

8. (a) Figure out a way to predict each numerator (p_n) just by looking at the previous two numerators (p_{n-1} and p_{n-2}) and the current term (a_n).
   (b) Figure out how to predict each denominator from the previous two denominators and the current term.

9. Calculate the "criss-cross" products $p_{n-1}q_n - p_nq_{n-1}$. What do you notice? Challenge: prove it.

10. Do the decimal values of the convergents increase or decrease?

11. Some continued fractions can be evaluated more simply: Find $[2; 4, 4, 4, 4, 4, \ldots]$ and $[2; 1, 4, 1, 4, 1, 4, \ldots]$

5 Writing Numbers as Continued Fractions

12. Write the following numbers as continued fractions:

    (a) $\frac{19}{13}$

    (b) $\frac{57}{67}$

    (c) $\sqrt{3}$

    (d) π

    (e) e

13. Can any number be written as a continued fraction?

6 Why do Continued Fractions Rock?

14. The true nature of a number revealed:
    (a) How can you tell if a number will have a finite or an infinite continued fraction expansion?
    (b) How can you tell if a number will have a repeating or non-repeating continued fraction expansion?

15. The number e:

    The decimal expansion of e is underwhelming: $2.71828182845905\ldots$

    Look at the continued fraction expansions of e and related expressions:

Number	Continued Fraction
e	$[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1 \ldots]$
$\frac{e-1}{e+1}$	$[0; 2, 6, 10, 14, \ldots]$
$\frac{e^2-1}{e^2+1}$	$[0; 1, 3, 5, 7, 9, 11, \ldots]$
$\sqrt{e}$	$[1; 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, 1, 1, 17, 1, 1, \ldots]$
$\sqrt[3]{e}$	$[1; 2, 1, 1, 8, 1, 1, 14, 1, 1, 20, 1, 1, \ldots]$
$\sqrt[4]{e}$	$[1; 3, 1, 1, 11, 1, 1, 19, 1, 1, 27, 1, 1, \ldots]$
$\sqrt[5]{e}$	$[1; 4, 1, 1, 14, 1, 1, 24, 1, 1, 34, 1, 1, \ldots]$

    What will the continued fraction of $\sqrt[10]{e}$ be?

16. Here are some other representations of e:

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cdots}}}}$$

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}}$$

17. Square roots. Here are the continued fraction expansions for some square roots. What patterns do you see?

$\sqrt{n}$	Continued Fraction				
$\sqrt{1}$	$[1]$	$\sqrt{41}$	$[6;\overline{2,2,12}]$	$\sqrt{81}$	$[9]$
$\sqrt{2}$	$[1;\overline{2}]$	$\sqrt{42}$	$[6;\overline{2,12}]$	$\sqrt{82}$	$[9;\overline{18}]$
$\sqrt{3}$	$[1;\overline{1,2}]$	$\sqrt{43}$	$[6;\overline{1,1,3,1,5,1,3,1,1,12}]$	$\sqrt{83}$	$[9;\overline{9,18}]$
$\sqrt{4}$	$[2]$	$\sqrt{44}$	$[6;\overline{1,1,1,2,1,1,1,12}]$	$\sqrt{84}$	$[9;\overline{6,18}]$
$\sqrt{5}$	$[2;\overline{4}]$	$\sqrt{45}$	$[6;\overline{1,2,2,2,1,12}]$	$\sqrt{85}$	$[9;\overline{4,1,1,4,18}]$
$\sqrt{6}$	$[2;\overline{2,4}]$	$\sqrt{46}$	$[6;\overline{1,3,1,1,2,6,2,1,1,3,1,12}]$	$\sqrt{86}$	$[9;\overline{3,1,1,1,8,1,1,1,3,18}]$
$\sqrt{7}$	$[2;\overline{1,1,1,4}]$	$\sqrt{47}$	$[6;\overline{1,5,1,12}]$	$\sqrt{87}$	$[9;\overline{3,18}]$
$\sqrt{8}$	$[2;\overline{1,4}]$	$\sqrt{48}$	$[6;\overline{1,12}]$	$\sqrt{88}$	$[9;\overline{2,1,1,1,2,18}]$
$\sqrt{9}$	$[3]$	$\sqrt{49}$	$[7]$	$\sqrt{89}$	$[9;\overline{2,3,3,2,18}]$
$\sqrt{10}$	$[3;\overline{6}]$	$\sqrt{50}$	$[7,\overline{14}]$	$\sqrt{90}$	$[9;\overline{2,18}]$
$\sqrt{11}$	$[3;\overline{3,6}]$	$\sqrt{51}$	$[7;\overline{7,14}]$	$\sqrt{91}$	$[9;\overline{1,1,5,1,5,1,1,18}]$
$\sqrt{12}$	$[3;\overline{2,6}]$	$\sqrt{52}$	$[7;\overline{4,1,2,1,4,14}]$	$\sqrt{92}$	$[9;\overline{1,1,2,4,2,1,1,18}]$
$\sqrt{13}$	$[3;\overline{1,1,1,1,6}]$	$\sqrt{53}$	$[7;\overline{3,1,1,3,14}]$	$\sqrt{93}$	$[9;\overline{1,1,1,4,6,4,1,1,1,18}]$
$\sqrt{14}$	$[3;\overline{1,2,1,6}]$	$\sqrt{54}$	$[7;\overline{2,1,6,1,2,14}]$	$\sqrt{94}$	$[9;\overline{1,2,3,1,1,5,1,8,1,5,1,1,3,2,1,1$
$\sqrt{15}$	$[3;\overline{1,6}]$	$\sqrt{55}$	$[7;\overline{2,2,2,2,14}]$	$\sqrt{95}$	$[9;\overline{1,2,1,18}]$
$\sqrt{16}$	$[4]$	$\sqrt{56}$	$[7;\overline{2,14}]$	$\sqrt{96}$	$[9;\overline{1,3,1,18}]$
$\sqrt{17}$	$[4;\overline{8}]$	$\sqrt{57}$	$[7;\overline{1,1,4,1,1,14}]$	$\sqrt{97}$	$[9;\overline{1,5,1,1,1,1,1,1,5,1,18}]$
$\sqrt{18}$	$[4;\overline{4,8}]$	$\sqrt{58}$	$[7;\overline{1,1,1,1,1,1,14}]$	$\sqrt{98}$	$[9;\overline{1,8,1,18}]$
$\sqrt{19}$	$[4;\overline{2,1,3,1,2,8}]$	$\sqrt{59}$	$[7;\overline{1,2,7,2,1,14}]$	$\sqrt{99}$	$[9;\overline{1,18}]$
$\sqrt{20}$	$[4;\overline{2,8}]$	$\sqrt{60}$	$[7;\overline{1,2,1,14}]$	$\sqrt{100}$	$[10]$
$\sqrt{21}$	$[4;\overline{4,1,1,2,1,1,8}]$	$\sqrt{61}$	$[7;\overline{1,4,3,1,2,2,1,3,4,1,14}]$		
$\sqrt{22}$	$[4;\overline{1,2,4,2,1,8}]$	$\sqrt{62}$	$[7;\overline{1,6,1,14}]$		
$\sqrt{23}$	$[4;\overline{1,3,1,8}]$	$\sqrt{63}$	$[7;\overline{1,14}]$		
$\sqrt{24}$	$[4;\overline{1,8}]$	$\sqrt{64}$	$[8]$		
$\sqrt{25}$	$[5]$	$\sqrt{65}$	$[8;\overline{16}]$		
$\sqrt{26}$	$[5;\overline{10}]$	$\sqrt{66}$	$[8;\overline{8,16}]$		
$\sqrt{27}$	$[5;\overline{5,10}]$	$\sqrt{67}$	$[8;\overline{5,2,1,1,7,1,1,2,5,16}]$		
$\sqrt{28}$	$[5;\overline{3,2,3,10}]$	$\sqrt{68}$	$[8;\overline{4,16}]$		
$\sqrt{29}$	$[5;\overline{2,1,1,2,10}]$	$\sqrt{69}$	$[8;\overline{3,3,1,4,1,3,3,16}]$		
$\sqrt{30}$	$[5;\overline{2,10}]$	$\sqrt{70}$	$[8;\overline{2,1,2,1,2,16}]$		
$\sqrt{31}$	$[5;\overline{1,1,3,5,3,1,1,10}]$	$\sqrt{71}$	$[8;\overline{2,2,1,7,1,2,2,16}]$		
$\sqrt{32}$	$[5;\overline{1,1,1,10}]$	$\sqrt{72}$	$[8;\overline{2,16}]$		
$\sqrt{33}$	$[5;\overline{1,2,1,10}]$	$\sqrt{73}$	$[8;\overline{1,1,5,5,1,1,16}]$		
$\sqrt{34}$	$[5;\overline{1,4,1,10}]$	$\sqrt{74}$	$[8;\overline{1,1,1,1,16}]$		
$\sqrt{35}$	$[5;\overline{1,10}]$	$\sqrt{75}$	$[8;\overline{1,1,1,16}]$		
$\sqrt{36}$	$[6]$	$\sqrt{76}$	$[8;\overline{1,2,1,1,5,4,5,1,1,2,1,16}]$		
$\sqrt{37}$	$[6;\overline{12}]$	$\sqrt{77}$	$[8;\overline{1,3,2,3,1,16}]$		
$\sqrt{38}$	$[6;\overline{6,12}]$	$\sqrt{78}$	$[8;\overline{1,4,1,16}]$		
$\sqrt{39}$	$[6;\overline{4,12}]$	$\sqrt{79}$	$[8;\overline{1,7,1,16}]$		
$\sqrt{40}$	$[6;\overline{3,12}]$	$\sqrt{80}$	$[8;\overline{1,16}]$		

7 Why are Continued Fractions Useful?

18. List a few fractions that approximate π.

19. Which approximation for π do you prefer: $\dfrac{22}{7}$ or $\dfrac{314}{100}$. Why?

20. A fraction is a *best approximation* to a number if there is no rational approximation as close or closer with as small or smaller a denominator.

    Here are consecutive best approximations of π:

$$\frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}, \frac{179}{57}, \frac{201}{64}, \frac{223}{71}, \frac{245}{78}, \frac{267}{85}, \frac{289}{92}, \frac{311}{99}, \frac{333}{106}, \frac{355}{113}, \frac{52163}{16604}, \frac{52518}{16717}$$

    Here are some convergents of π

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \ldots$$

    What do you notice about the best approximations and the convergents?

21. The *mediant* of two fractions is the fraction you get by adding together numerators and denominators. For example, $\dfrac{3}{1} \oplus \dfrac{22}{7} = \dfrac{25}{8}$.

    Show that the mediant of two positive fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ must lie in between the two fractions.

22. Start with $\dfrac{1}{0}$ and $\dfrac{3}{1}$. Take the mediant, and then take the mediant of the result with $\frac{3}{1}$, and the mediant of that result with $\frac{3}{1}$, etc. Fill in the chart with your results. What do you notice? Try the same thing with $\dfrac{3}{1}$ and $\dfrac{22}{7}$.

A	B	$A \oplus B$	Decimal value of $A \oplus B$
$\frac{1}{0}$	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		
	$\frac{3}{1}$		

A	B	$A \oplus B$	Decimal value of $A \oplus B$
$\frac{3}{1}$	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		
	$\frac{22}{7}$		