CHMC Advanced: Covering Problems

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1 Covering a cube

An **n-digit** is a string of the form $a_1a_2...a_n$, where each of the a_is are 0 or 1. For example

- these are all possible 2-digits: $00, 01, 10, 11$,
- these are some possible 3-digits: $001, 010, 110$,
- this is a 4-digit: 0100.

The **n-cube** is the collection of all possible n-digits for a given n.
The distance between two digits $a_1...a_n$ and $b_1...b_n$ is the quantity

$$d(a_1...a_n, b_1...b_n) = |a_1 - b_1| + ... + |a_n - b_n|;$$

intuitively, the distance between two digits is the number of places that the digits differ. For example,

- $d(010, 011) = |0 - 0| + |1 - 1| + |0 - 1| = 1$,
- $d(1111, 0000) = 4$,
- $d(10, 0110)$ is undefined, because the digits have different lengths.

The **unit disk** $D(a_1...a_n)$ in an n-cube, centered at the digit $a_1...a_n$, is the set

$$D(a_1...a_n) = \{b_1...b_n : d(a_1...a_n, b_1...b_n) \leq 1\}.$$

For example, the following are all unit disks:

- $D(01) = \{01, 00, 11\}$,
- $D(0110) = \{0110, 1110, 0010, 0100, 0111\}$.

What is the minimum number of closed disks required to cover the 2-cube? The 3-cube? The n-cube?

2 Covering a sphere

An n-**vector** is a sequence of n real numbers, usually thought of geometrically as an arrow drawn in space. For two 2-vectors, their **angle** (in degrees) is the usual geometric angle between two vectors in the plane. Two n-vectors v_1, v_2 that are not parallel determine a plane P, and we can think of v_1, v_2 as being 2-vectors on P. With this picture in mind, the angle between the n-vectors v_1, v_2 is the angle between their 2-vector representations on P.

What is the maximum number of (non-zero) 2-vectors you can choose such that the angle between any two of them is between $90°$ and $180°$? In other words, we want a collection of 2-vectors $\{v_1, ..., v_N\}$ such that the angle between any v_i, v_j is between $90°$ and $180°$; what is the largest number N can be? What is the maximum number of 3-vectors you can collect such that, pairwise, their angles are between $90°$ and $180°$? What about n-vectors?

How can this problem be cast as a "covering problem"?