“Insert Sum Here” and the Calkin–Wilf Tree

1 Introduction: “Insert Sum Here”

A recent worksheet at Chapel Hill’s Julia Robinson Mathematics Festival for 2019, Joshua Zucker’s *Insert Sum Here* (henceforth “ISH”), presented an open-ended exercise which introduced a sequence that had a surprising amount of structure. That worksheet appears to be unavailable online presently, but some background on the worksheet is provided in a Numberplay entry of *The New York Times*’ Wordplay blog [1]. There is also a multi-part series of posts on this topic in Brent Yorgey’s *Math Less Traveled* blog, at [6], [11], [10], [9], [8], [4], [14], [5], [14], [15], [12], and [13], with series index at [7].

We begin by defining the ISH array, continue by exploring its properties, and we later introduce the related Calkin–Wilf sequence.

To begin, consider the following infinite sequence of rows, whose recursive construction is analogous to that of Pascal’s Triangle:

$$\begin{array}{ccccccccccccccccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 2 & 3 & 1 \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The pattern is suggested by the title: to form the next row, take all the entries of the previous row, insert spaces between consecutive terms, then fill each space with the sum of the two adjacent terms. Explicitly, let $a(j,k)$ denote the jth entry in the kth row, and take following recursive definitions: for all positive integers j, k, define

$$a(1,1) := 1 \tag{1.1}$$
$$a(1,2) := 1 \tag{1.2}$$
$$a(j+1,2k-1) := a(j,k) \tag{1.3}$$
$$a(j+1,2k) := a(j,k) + a(j,k+1). \tag{1.4}$$

For ordered pairs (j,k) such that $a(j,k)$ cannot be computed from (1.1)–(1.4), we say $a(j,k)$ is undefined.

Our goal is be to explore the structure of the ISH sequence and related mathematical structures.

2 Basic Properties of the ISH Sequence

2.1. Let R_m denote the mth row of this sequence. In (1), we are given R_1 through R_5 explicitly. Produce R_6 explicitly. Can you also compute R_7?

2.2. Let m be a positive integer. How many elements are in R_m? That is, what is $|R_m|$? Consider whether you can produce a recursive formula, which depends on values of $|R_j|$ for $j < m$, as well as whether you can produce a closed-form expression for $|R_m|$ depending on m alone.

Note: We are asking for how many entries lie in R_m, not how many *distinct* entries there are. For example, $|R_1| = 2$, and $|R_4| = 9$, even though both rows include duplicate entries.

2.3. Let S_m denote the sum of all the entries in R_m. Compute S_m. As in Exercise #2.2), consider both recursive and closed-form solutions.

2.4. Let m be any positive integer, and let n be a positive integer such that both $a(m,n)$ and $a(m,n+1)$ appear in R_m. Prove that $\gcd(a(m,n), a(m,n+1)) = 1$. That is, prove that any two consecutive terms in a given row are relatively prime.

2.5. For each positive integer m, what is the largest element of R_m?

2.6. For each positive integer m, how many times does m appear in R_m? Does this number change if we consider how many times m appears in R_j, where j is any positive integer?

2.7. Consider the following 2×2 submatrices of the ISH sequence:

Define the *determinant* of a 2×2 matrix by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc. \tag{2.1}$$

What do you notice about these 2×2 determinants? Can you prove whether this always holds for any determinant formed from a 2×2 submatrix of the ISH sequence?

2.8. Extend Exercise #2.7 by considering 3×3, 4×4, and in general, $n \times n$ determinants. Form a conjecture about the value for any such higher order determinant. Can you prove it?

3 Hyperbinary Representations of Positive Integers

In this section, we consider the number of *hyperbinary* representations of a positive integer n, then show the connections between this number and the ISH sequence.

Definition 3.1. Let n be a nonnegative integer. A *hyperbinary* expansion of n is a sum of powers of two, used at most twice, that sum to n. For a given nonnegative integer n, we let $h(n)$ denote the number of distinct hyperbinary representations of n. By convention, $h(0) := 1$.

Example 3.2. Consider the following examples:

- $h(2) = 2$: we have the two hyperbinary decompositions $2 = 2^1 = 2^0 + 2^0$, and no other hyperbinary decompositions of 2 are possible.
- $h(6) = 3$, since $6 = 4+2 = 4+1+1 = 2+2+1+1$, and no other hyperbinary decompositions of 6 are possible.
- $h(7) = 1$, since $7 = 4+2+1$ is the unique hyperbinary representation of 7.
- $h(10) = 5$, since $10 = 8+2 = 8+1+1 = 4+4+2 = 4+4+1+1 = 4+2+2+1+1$, with no other hyperbinary decompositions of 10 possible.

3.1. Compute $h(n)$ for small values of n.

3.2. What is the relationship between the hyperbinary counting function h and the ISH sequences of the previous sections?

3.3. Prove that for all positive integers n, $h(2^n - 1) = 1$.

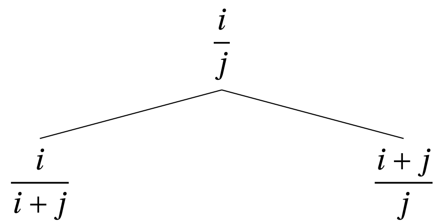
3.4. Let n be a nonnegative integer. Prove

$$h(2n+1) = h(n) \tag{3.1}$$

$$h(2n+2) = h(n) + h(n+1). \tag{3.2}$$

4 The Calkin–Wilf Tree

Closely related to the ISH sequences is the *Calkin–Wilf tree*. This is a graph-theoretic tree with positive rationals of the form $\frac{i}{j}$ at each node, where i, j are positive integers, and the fraction is in lowest terms. (That is, $\gcd(i, j) = 1$.) Then each such entry $\frac{i}{j}$ has two "children" given by the following formulas:



That is,

$$\text{left}\left(\frac{i}{j}\right) := \frac{i}{i+j} \tag{4.1}$$

$$\text{right}\left(\frac{i}{j}\right) := \frac{i+j}{j}. \tag{4.2}$$

These elements $\frac{i}{i+j}$ and $\frac{i+j}{j}$ are the left and right children of $\frac{i}{j}$, respectively. Each of these children will itself have a left and right child, too, and the tree extends down with infinitely many rows.

Via this rule, the first several rows of the Calkin–Wilf tree are given below:

Note: Exercises #4.4–4.6 are taken from Section 1 of Calkin and Wilf [3].

4.1. Produce the first three to five rows of the Calkin–Wilf tree.

4.2. Describe a relationship between the ISH sequence and the Calkin–Wilf tree.

4.3. Above, we have the rules for how to form the right and left children of an entry $\frac{i}{j}$ in the Calkin–Wilf tree. Assuming $\frac{r}{s}$ is a positive rational number with $\frac{r}{s} \neq 1$, what are the possible "parents" of $\frac{r}{s}$? For a particular positive rational of the form $\frac{r}{s}$, can we determine which formula is correct?

4.4. Let $\frac{i}{j}$ be a positive rational number appearing in the Calkin–Wilf tree generated by $\frac{1}{1}$ and successive applications of the formulas (4.1) and (4.2). Prove that $\frac{i}{j}$ is in lowest terms. That is, show $\gcd(i, j) = 1$.

4.5. Prove that every positive rational number appears at least once in the Calkin–Wilf tree.

4.6. Prove that every positive, rational q appears *at most once* in the Calkin–Wilf tree. Combining this result with that of Exercise #4.5, show how this provides an explicit enumeration of the positive rational numbers. (That is, the set of positive rationals is *countably infinite*, meaning it is in *one-to-one correspondence* or *bijection* with the set of all positive integers.)

4.7. Let q be a rational number. Then a *continued fraction expansion* for q is an expression of the form

$$[a_0; a_1, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}, \tag{4.3}$$

where a_0 is an integer, $a_1, \ldots, a_n$ are *positive* integers, and n is a nonnegative integer. Further, we do not let $a_n = 1$ unless $n = 1$ and $a_0 = 0$, since otherwise $[a_0; a_1, \ldots, a_{n-1}, 1] = [a_0; a_1, \ldots, a_{n-1} + 1]$.

Compute the continued fraction expansions for the first 3–5 rows of the Calkin–Wilf tree.

Note: This exercise becomes a bit easier if you can first solve Exercise #4.8, especially if trying to compute the continued fractions for rows deep into the Calkin–Wilf tree.

4.8. Let $q := \frac{i}{j}$ be a positive, rational number reduced to lowest terms. Further, let the continued fraction expansion for q be $[a_0; a_1, \ldots, a_n]$. What are the respective continued fraction expansions for the left and right children of q?

4.9. Let q be a positive rational with continued fraction expansion $[a_0; a_1, \ldots, a_n]$. Provide a complete characterization to determine the number of the unique row in the Calkin–Wilf tree where q appears.

Note: This exercise is inspired by Sections 5 and 9 in Bates, Bunder, and Tognetti [2]. That article goes into *much* more detail about identifying precisely where a positive rational q appears in the Calkin–Wilf tree.

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