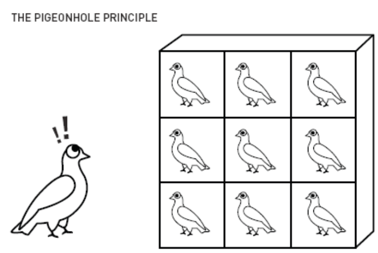
Pigeonhole Principle

1 Warm-up problems

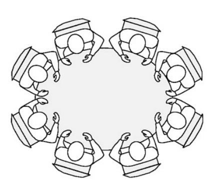
1. I own 7 pairs of socks and each pair is a different color. If all 14 socks are loose in the dryer, how many will I have to pull out to guarantee that I get at least two of the same color?
2. A group of 7 children order ice cream cones from a store that has the following flavors: vanilla, chocolate, strawberry, mint chocolate chip, coconut. Show that at least two of them get the same flavor.
3. There are over 10 million people in North Carolina. Show that at least two of them have the exact same number of hairs on their head. (The average number of hairs on a human head is 100,000.)

2 Pigeonhole problems

> **Pigeonhole Principle:**
> If you put more than n pigeons into n pigeon holes, at least one pigeon hole must contain more than one pigeon.



4. Eight chairs are set around a circular table. On the table are name placards for eight guests. After the guests are seated, it is discovered that none of them are in front of their own names. Show that the table can be rotated so that at least two guests are simultaneously correctly seated.



5. Given 12 integers, show that two of them can be chosen whose difference is divisible by 11.
6. (a) How many cards must be selected from a SET deck to guarantee that at least three cards of the same shape are chosen? (There are 81 SET cards total, and three different shapes: diamond, oval, and squiggle.)
   (b) How many cards must be selected to guarantee that at least three squiggles are selected?

7. Sixteen boxes of chocolate are for sale at the store. The chocolates are of three different kinds (dark, milk chocolate, and white chocolate), and all chocolates in a box are of the same kind. You want to buy 6 boxes of chocolates to give to your 6 cousins, but you want to give them all the same kind of chocolate so there won't be any squabbling. Is this necessarily possible?

Pigeonhole Principle, Advanced Version:

(a) If you put more than $2n$ pigeons into n pigeon holes, at least one pigeon hole must contain more than 2 pigeons.

(b) If you put more than $5n$ pigeons into n pigeon holes, at least one pigeon hole must contain more than 5 pigeons.

(a) (c) If you put more than kn pigeons into n pigeon holds, at least one pigeon hole must contain more than k pigeons.

8. How many students do you need to have in a class to guarantee that at least 4 students will have birthdays in the same month?

3 Pigeonhole problems with twists

9. Show that in any group of five people, there are two who have an identical number of friends within the group. (Friendship is mutual – if A is B's friend, then B is A's friend.)

10. Of 40 children seated at a round table, more than half are girls. Prove that there are two girls who are seated diametrically opposite each other.

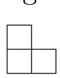
11. Several soccer teams enter a tournament in which each team plays every other team exactly once. Show that, at any moment during the tournament, there will be two teams which have played, up to that moment, the same number of games.

12. Prove that there exist two powers of 2 whose difference is a multiple of 2019. (A power of 2 is a number that can be reached by multiplying 2 by itself a bunch of times, like 2, 4, 8, 16, 32, etc..)

4 Geometry Problems

13. Fifty-one points are scattered within a square of side length one meter. Show that at least 3 of the points can be covered with a square of side length 20 cm.

14. (a) What is the largest number of squares on an 8×8 checkerboard which can be colored green, so that in any arrangement of three squares (a "tromino" as drawn below), at least one square is not colored green? (The tromino may be appear as in the figure or it may be rotated through some multiple of 90 degrees.)



(b) What is the smallest number of squares which can be colored green, so that in any tromino at least one square is colored green?

15. Show that an equliateral triangle cannot be covered completely by two smaller equilateral triangles.

16. (a) Given five points in a square of side length 1, show that two of the points must be no more than $\sqrt{2}/2$ apart.

    (b) Given five points in an equilateral triangle of side length 2, show that two of the points must be no further than 1 apart.

    (c) Given nine points in a square of side length 2, show that three of the points must form a triangle whose area is not more that 1/2.

5 Hard Problems

17. Suppose we are given 10 different numbers from 1, 2, 3, . . . 99. Prove that there exists two disjoint subsets with the same sum. For example, if we are given the ten numbers $\{1, 2, 7, 11, 23, 31, 35, 48, 83, 91\}$ then

$$2 + 11 + 23 = 35 + 1$$

18. Given eight different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference. (The pairs may overlap – that is, two pairs or all three pairs may contain the same integer.)

19. Prove that among any six people, there are either at least three people who all know each other or at least three people who are all strangers to each other. (Assume that if person A knows person B, then person B also knows person A.)

20. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.

21. Prove that you can choose a subset of ten given integers such that their sum is divisible by 10.

22. Given 11 different positive integers, none greater than 20, prove that two of these can be chosen, one of which divides the other.

These problems are from *Mathematics Circles: the Russian Experience* by Fromkin, Genkin, and Itenberg and from *A Decade of the Berkeley Math Circle* by Stankova and Rike, and from UNC's Problem Solving Seminar.