CHMC Advanced Section: Loops

September 14, 2019

1 Introduction

The purpose of this worksheet is to introduce the notion of the fundamental group. The fundamental group is an algebraic object that encodes a lot of topological/geometric information about a space in question. In particular, if two spaces are topologically "the same", then they will have the same fundamental group. This lets us use this object as a way to distinguish between spaces: if their fundamental groups are different, the spaces must be topological distinct. One instance where this pops up is in distinguishing between a donut and a sphere: intuitively these are different objects, but mathematically how might we say so? The fundamental group gives us a way.

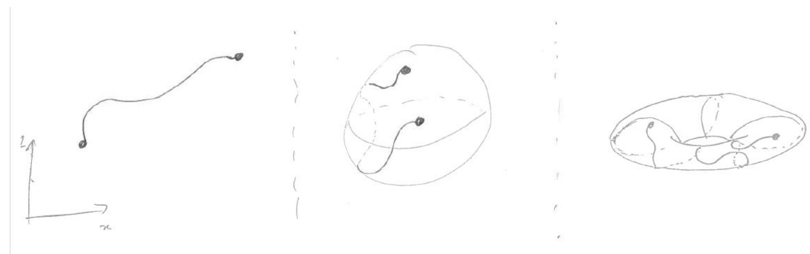
In this worksheet, we'll introduce some of the preliminary mathematical objects needed to understand what the fundamental group is all about. In practice, computing a space's fundamental group can be very involved, so our focus will be on building intuition about this object.[1]

2 Homotopy equivalence

The basic objects that will come up throughout this worksheet are paths, loops, and homotopies; this section describes what these are.

Definition 1. A *path* is a continuous curve from one point to another. Formally, a path f is a continuous function from the interval $[0, 1]$ to a space X. The starting point of the path is the point $f(0)$, and the end point is $f(1)$, and intermediate points along the path are denoted $f(t)$ for $0 < t < 1$.

[1] All of the figures were either sketched by hand, or taken from Allen Hatcher's book on Algebraic Topology. It's usually pretty clear which is which.

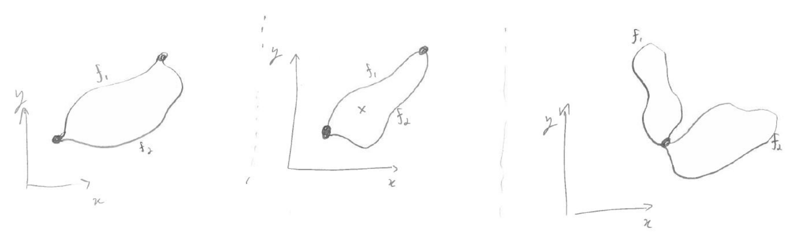


Definition 2. A *loop* is a path that starts and ends at the point, i.e. $f(0) = f(1)$.

Definition 3. If f_0 and f_1 are two paths (or loops), then we say they are *homotopic* if there is a continuous deformation of f_0 to f_1.

In practice, we think of homotopies as stretching, bending, and deforming a loop without cutting it.

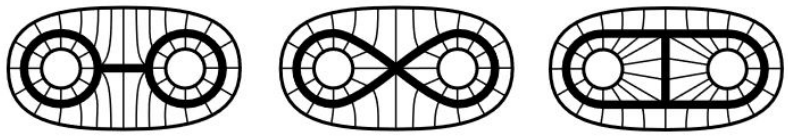
Exercise 1. Which of the following pairs of loops are homotopic?



To define homotopies of spaces, we essentially take the same definition as we had for paths (and loops), with a few changes.

Definition 4. Intuitively, two spaces (shapes, regions, etc.) should be homotopy equivalent if they can be continuously deformed into one another without cutting or tearing the shape. Formally, if $f\colon X \to Y$ is a map from X to Y, and $g\colon Y \to X$ is a map from Y to X, then we say X and Y are *homotopy equivalent* if $f \circ g$ and $g \circ f$ are homotopic to the identity map.

The formal definition is good for proving things, whereas the intuitive "definition" is helpful for actually homotoping spaces. Let's see an example:



Exercise 2. All of the above spaces are homotopic to a figure 8; why?

Exercise 3. Show that the punctured plane is homotopy equivalent to the unit circle.

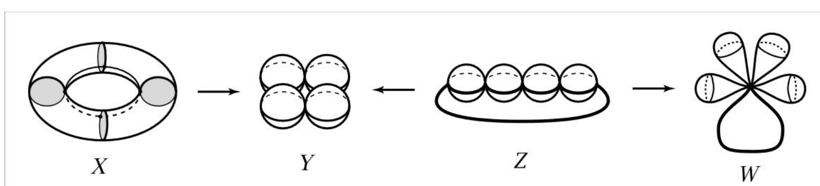
Exercise 4. Generalize this last exercise: show that the plane with n points removed is homotopy equivalent to a "bouquet" of n circles. Sketch this bouquet of circles.

Another family of examples involve spheres and loops.

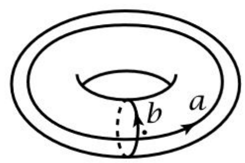
Exercise 5. Show that the above three spaces are homotopy equivalent.

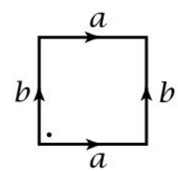
The last two exercises explored the "stretching and bending" aspect of homotopies. Another important aspect not really highlighted above is the "moving around" of spaces. The next exercise explores this aspect.

Exercise 6. Show that each of the spaces X, Y, Z, and W are homotopy equivalent.



Another space topologists love to study is the torus, which can be constructed by taking a square (as drawn) below and adjoining parallel sides.



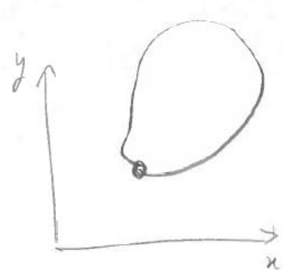


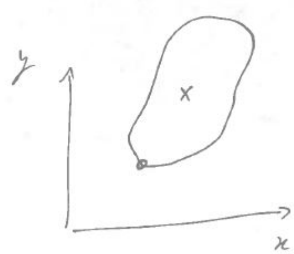
Exercise 7. Show that if we remove a point from the torus, the resulting shape is homotopic to a bouquet of two circles.

Definition 5. A homotopy that collapses a region to a point (in a continuous way) is called a *contraction.* A space is *contractible* if it is homotopy equivalent to a point.

Something to note is that contractibility depends on the ambient space of the loops/spaces we're interested in.

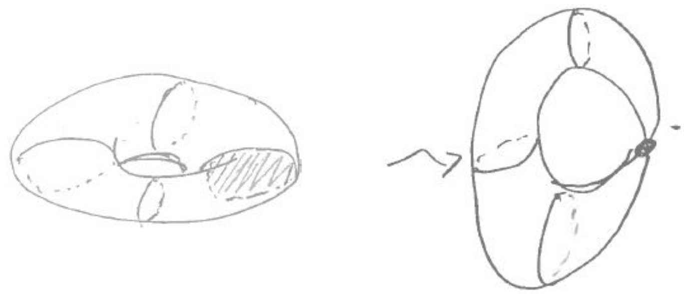
In the figure below, the loop on the left is contractible in the plane while the loop on the right is not. The reason for this is that, to contract the loop on the right, we'd need to pass over the removed point. But if our path passed over that point, we would no longer have a loop contained in our space, so the transformation would not be continuous.





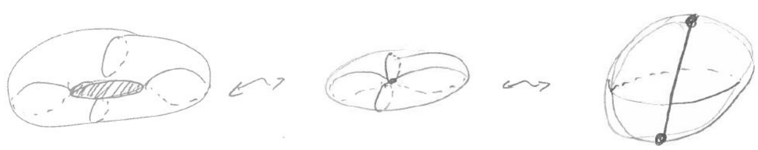
Exercise 8. Show that if f and g are two paths with the same endpoints, then $f \sim g$ if and only if the region enclosed by the trajectories of f and g is contractible.

Exercise 9. Show that the torus with a disc glued into the tube, as drawn below, is homotopic to the crescent shape we saw earlier.

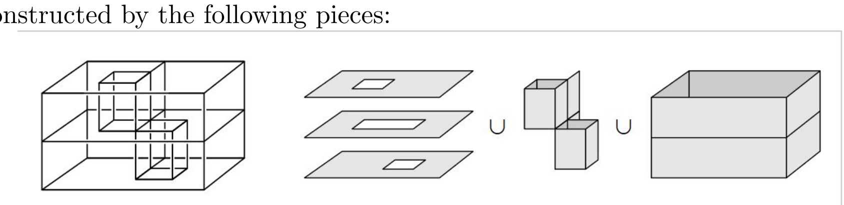


Earlier, we saw that the crescent moon shape was homotopic to a sphere with a loop attached. Interestingly, we can get the same object by gluing a disc into a torus in a different way.

Exercise 10. Show that the torus with a disc glued into its center (the donut hole) is homotopic to the other shapes drawn below.



An interesting example of a contractible space is the "house with two rooms", constructed by the following pieces:



On the left is a wireframe sketch of the final house, and the three sets on the right comprise the ceilings and floors, walls, and tunnels (respectively) of the house.

Exercise 11. Show that this house with two rooms is contractible. It may help to try constructing this space with play-doh.

Exercise 12. * What might a house with n-rooms look like?

3 The fundamental group

The fundamental group of a space is the collection of all loops starting (and ending at the same point). The fundamental group encodes quite a bit of topological, and sometimes geometric, information about the space in consideration.

We'll start by looking at some properties of loops, and homotopies of loops, after which we'll define the fundamental group and look at a few examples.

Definition 6. If f and g are two loops (based at a point p), then their concatenation is the loop that first follows f, and then follows g. Explicitly, we define the path via

$$(f \star g)(t) := \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Exercise 13. Make sense of this definition.

Our next goal is to show that the concatenation of loops is associative, i.e. that $(f \star g) \star h \sim f \star (g \star h)$. In practice, this means that if we concatenate two loops f and g, and then concatenate that loop with h, we'd get the same loop (up to homotopy) as if we had first concatenated g and h, and then f on the left.

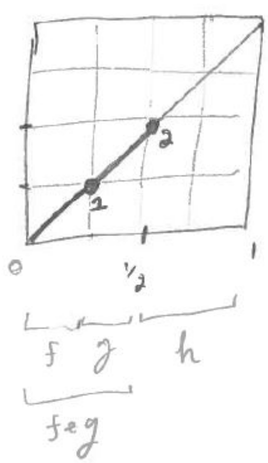
In theory, we'd need to show that we can reparametrize the concatenations in such a way to make the left and right hand sides agree. In practice, we can get away with sketching the parametrization as a square, and working with the homotopies directly. Let's see how this works:

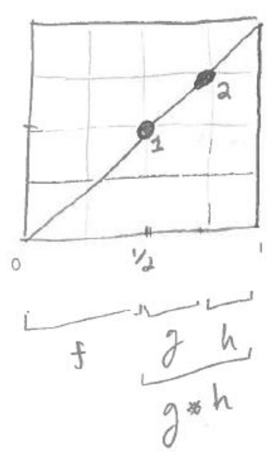
We have the two curves

$$((f \star g) \star h)(t) = \begin{cases} (f \star g)(2t), & 0 \leq t \leq \frac{1}{2}, \\ h(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$\text{and} \quad (f \star (g \star h))(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2}, \\ (g \star h)(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}.$$

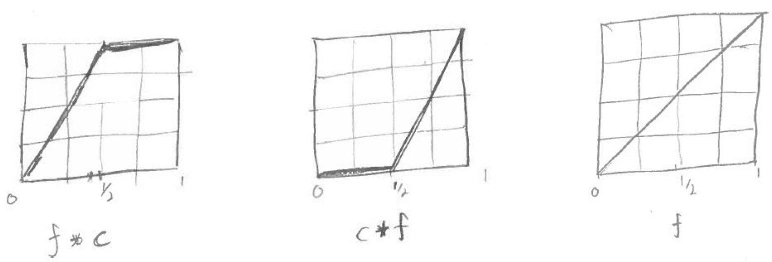
These two parametrizations can be represented by the following two squares:





Point 1 indicates where the loop f ends and the loop g begins, and point 2 indicates where g ends and h begins. The x-axis represents the value of t, where along the path we are. By homotoping the parametrization curve, we can "slide" points 1 and 2 from the left curve up and over so that we end up with the configuration on the right. This homotopy corresponds to a reparametrization of the loop's domain.

This strategy can also be used to show that if c is a constant loop (for all time t, the loop stays at the same spot), and f is another loop, then $c \star f \sim f$ and $f \star c \sim f$. In this case, the relevant figures are



Exercise 14. Prove this last assertion, that $c \star f \sim f$ and $f \star c \sim f$.

We can also "undo" loops.

Definition 7. If f is a loop, define $\bar{f}$ to be the loop that traverses f backwards.

Exercise 15. Show that the parametrization for the loop $\bar{f}$ can be represented as a square with a diagonal from the top left corner to the bottom right corner.

Exercise 16. Show that $f \star \bar{f}$ is homotopic to a constant path.

The final piece we need is the fact that paths, with respect to homotopy, form an equivalence relation. This means that

1. every loop f is homotopic to itself (written $f \sim f$),
2. if f is homotopic to g, then g is homotopic to f (written $f \sim g$ implies $g \sim f$), and vice versa, and
3. if f is homotopic to g, and g is homotopic to h, then f is homotopic to h (written $f \sim g$ and $g \sim h$ implies $f \sim h$).

Exercise 17. Prove the above three assertions.

Thus, "distinct" loops are completely determined by the homotopy class of all possible loops. This means that, in practice, if two loops are homotopic to one another, we consider them the same.

Definition 8. The *fundamental group* of a space X is the collection of all homotopy classes of loops, based at some point in X.

Because loops only depend on their homotopy class, just looking at homotopy classes of loops is a valid approach. Moreover, we saw that concatenating loops based at the same point produces a new loop at that point. This is why it's called the fundamental **group** and not just the fundamental set, say.

Exercise 18. Show that the fundamental group of the punctured plane is the same as the fundamental group of the unit circle. What do you think the fundamental group of the unit circle is?

If there is only one homotopy class of loops, so that all possible loops are homotopic to one another, we say the space has a trivial fundamental group.

Exercise 19. Show that if a space is contractible, then the fundamental group is trivial. In other words, show that all loops are homotopy equivalent to a constant path.

The converse need not be true, there are examples of spaces (surfaces) that have trivial fundamental group but are not contractible. The next few exercises examine this space and some of its relatives.

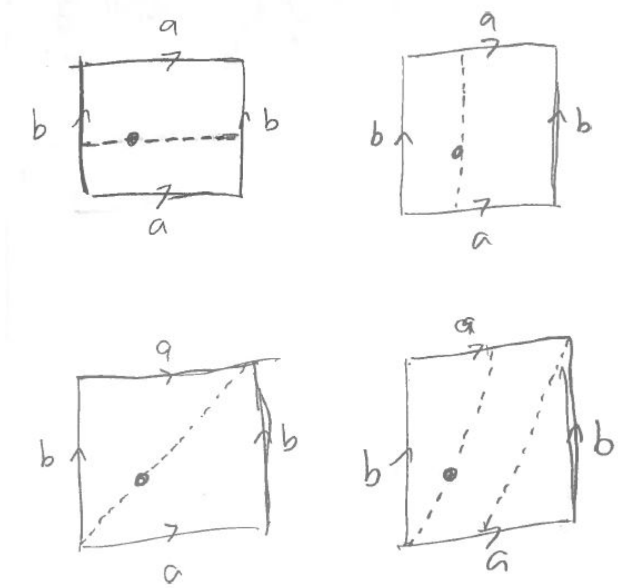
Exercise 20. Show that the fundamental group of the unit sphere is trivial, i.e. all loops are homotopic to a constant path.

Exercise 21. Show that the fundamental group of the sphere with a single point removed is trivial, whereas the fundamental group of a sphere with two points removed is not trivial.

Exercise 22. In general, show that the fundamental group of a sphere with n points removed is the same as the fundamental group of the plane with $n-1$ points removed.

Let's go back to working with the torus.

Exercise 23. Sketch what the following curves, drawn on a square, would look like on a torus.



Exercise 24. Show that if f and g are two loops on a torus, then $f \star g \sim g \star f$. Hint: try homotoping f and g into a combination of simpler curves.

This property, that the order in which pieces of a path may be traversed, doesn't hold in general.

Exercise 25. Consider the plane with two points, p_1 and p_2, removed. Let f_1 be a loop encircling p_1, and f_2 a loop encircling p_2. Argue that $f_1 \star f_2$ is not homotopic to $f_2 \star f_1$.

In this section, we weren't very precise about what the basepoint was for all of these loops. If the space X is *path-connected*, which means that any two points of X can be connected by a path, then the basepoint really doesn't matter.

Exercise 26. Suppose X is a path-connected space, let p and q be two points in X, and let γ be a path connecting p to q (so that γ is a continuous function, $\gamma(0) = p$, and $\gamma(1) = q$). Show that any loop f based at q can be "converted" to a loop based at p.