CHMC Advanced Group: MERLIN

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1 Introduction

In logic games and puzzles, there can be interesting mathematics at play underlying the activity. In this session, we will explore the game MERLIN, with the particular Magic Square invariant. In this, we explore matrices, modular arithmetic, and how to combine both in linear algebra.

2 Merlin's Magic

The game MERLIN is an electronic game with a variety of different sets of gameplays. In this, we will explore Magic Square.

The game consists of a 3×3 square board, with lights that toggle on or off. The labelling of each square is as follows

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The moves of the game involve pressing different squares, which toggle nearby squares on and off. To be precise, pressing square i will switch the on/off state of i and some of its neighbors. The corner squares $1, 3, 7, 9$ will toggle the squares that make up a 2×2 square that the corner is in. For instance, pressing square 1 will switch the states of squares $1, 2, 4, 5$, pressing square 3 will switch the states of $2, 3, 5, 6$ and similarly with the other corners. Pressing a middle edge square, $2, 4, 6, 8$ will toggle the whole edge on which it sits. So, pressing square 2 toggles $1, 2, 3$, pressing square 4 toggles $1, 4, 7$ and similarly for the other middle edge squares. Pressing the middle square, 5, will toggle itself and the adjacent four squares, that is squares $2, 4, 5, 6, 8$.

The goal of the game is, given an initial configuration, to press squares so that the middle square is off but all the other squares are on.

One way to represent the lights is through the use of 3×3 matrices. A 1 in a square i indicates that light i is on and 0 indicates that light i is off. For instance, if lights 3,5,7 are on, this is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

If you pressed square 2 from this configuration, this would flip the states of squares $1, 2, 3$, resulting in the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Exercise 2.1 What is the effect of pressing the same square twice? Consider the configuration given by $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Experiment with pressing a fixed square an even number of times or an odd number of times. What happens?

Exercise 2.2 Try solving the puzzle when the initial configuration is given by the matrix

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Record the moves you make. Can you solve it using fewer moves than you recorded?

Exercise 2.3 How about solving the puzzle from the initial configuration $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

3 $\mathbb{Z}_2$ and matrix operations

The group of two elements is called $\mathbb{Z}_2$, with elements $\{0, 1\}$. Addition and multiplication are given by the following rules

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

We can consider a matrix with entries in $\mathbb{Z}_2$, where matrix addition is given by addition of the corresponding entries in the matrix.

Exercise 3.1 What is the resulting matrix from

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = ?$$

What about adding these two matrices in the opposite order? Also, what happens when a matrix is added to itself?

From your observations in the previous section that toggling the same square an even number of times leaves a given configuration unchanged and toggling it an odd number of times changes it, we can relate picking a square to matrices with entries in $\mathbb{Z}_2$. Suppose that picking square i toggles square j. If square j is on, with value 1, picking square i turns j off, to 0. If square j is off, 0, picking square i turns j back on, 1.

Exercise 3.2 With the above observation in mind, what do you think the matrix u_1 for picking square 1 is? Consider the effect that picking square 1 has on all 9 squares. Do the same analysis for picking square 3, to determine the matrix u_3.

Exercise 3.3 Repeat the above exercise for picking square 2, for matrix u_2 and picking square 4, for matrix u_4.

Exercise 3.4 Repeat the above exercise for picking square 5, to determine the matrix u_5.

Exercise 3.5 Show that $u_1 + u_3 = u_3 + u_1$. Similarly, show that $u_2 + u_5 = u_5 + u_2$. Do you think it is the case that $u_i + u_j = u_j + u_i$ for $1 \leq i \leq j \leq 9$?

Exercise 3.6 What is the matrix that results from $u_1 + u_1$? How about $u_4 + u_4$? In general, is there a pattern for the matrix resulting in $u_i + u_i$ for $1 \leq i \leq 9$?

The previous two exercises show that picking square i and then j has the same effect as picking square j and then square i, and that picking the same square twice results in no change.

Exercise 3.7 What is the matrix resulting from $u_1 + u_7 + u_9 + u_7 + u_9$? Using the observations above, can you find a quick way to determine how to add a series of matrices when there are repeats (i.e., the sequence u_1, u_7, u_9, u_7, u_9 includes two instances of u_7 and two instances of u_9)?

Since the goal of the game is to start from a given configuration

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

and end at

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

by adding a combination of $u_1, u_2, \ldots, u_9$, we can ask for which initial configurations is this possible and what collection of moves are required. That is, do there exist $s_1, \ldots, s_9$, where $s_i \in \{0, 1\}$ for which $v_f = v_0 + s_1 \cdot u_1 + s_2 \cdot u_2 + \cdots + s_9 \cdot u_9$?

One alternative way of representing the moves u_i, initial configuration v_i and final configuration v_f, is by using column vectors. In this setting we see that $u_1, u_2, u_5, and v_f$ have the following representations

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } v_f = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then the questions stated above can be re-framed as given $v_0 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}$ where $a_i \in \{0, 1\}$, do there exist $s_1, \ldots, s_9$ where $s_i \in \{0, 1\}$ for which

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} + s_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \cdots + s_9 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} ?$$

This we can more compactly represent as $v_f = v_0 + As$ where $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{pmatrix}$ and A is the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

This matrix is special in that it has an inverse, A^{-1} such that $AA^{-1} = I = A^{-1}A$, where I is the matrix with 1's along the diagonal and 0's elsewhere. Therefore, for any $v_0 \neq v_f$,

$$v_f - v_0 \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ so } \quad A^{-1}(v_f - v_0) = A^{-1}As = s$$

so we can determine which moves we need to make to solve the puzzle. In this case,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Exercise 3.8 Suppose $v_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Using the method above, what moves are required to solve the game? This initial configuration is associated to the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Verify that the moves indicated from the solution work by trying them out to get the goal matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Exercise 3.9 Suppose that $v_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, corresponding to the initial configuration given by $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Verify that the solution you determine using linear algebra is indeed the solution by checking the associated moves.

4 Generalization

Suppose that instead of the lights turning on or off, they have several states they cycle through. For instance suppose that lights are off with value 0, red at value 1 and blue at value 2. Each time a square is chosen, the same moves apply as before, but the lights cycle through off, red, blue and back to off again, repeating.

For instance, a matrix might look like

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

and after pressing square one results in

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

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Exercise 4.1 For a given configuration, what happens if you press the same square 2, 3, or 4 times? Do you see a pattern?

There is a connection to the group $\mathbb{Z}_3$ of three elements $\{0, 1, 2\}$, which has the following addition and multiplication tables.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

We represent the possible moves using matrices with values in $\mathbb{Z}_3$ rather than $\mathbb{Z}_2$. The matrices for u_i from before are unchanged, however the matrix for A^{-1} does in fact change.

In this setting,

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

In this case, for any $v_0 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ with $a_i \in \{0, 1, 2\}$, or equivalently $v_0 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} \neq$ $v_f = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, we can express $v_f - v_0 = As$ where $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{pmatrix}$ with $s_i \in \{0, 1, 2\}$ not all zero.

Hence, $A^{-1}(v_f - v_0) = A^{-1}As = s$ and we can solve the system to determine which moves are required to transform the initial configuration to the target configuration.

Exercise 4.2 Suppose $v_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Using the method above, what moves are required to solve the game? This initial configuration is associated to the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Verify that the moves indicated from the solution work by trying them out to get the goal matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Exercise 4.3 Suppose that $v_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, corresponding to the initial configuration given by $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Verify that the solution you determine using linear algebra is indeed the solution by checking the associated moves.

Exercise 4.4 Suppose $v_0 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, corresponding to the initial configuration given by $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. Verify that the solution you determine using linear algebra is indeed the solution by checking the associated moves.

Exercise 4.5 Suppose that $v_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, corresponding to the initial configuration given by $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Verify that the solution you determine using linear algebra is indeed the solution by checking the associated moves.

It turns out that if instead of 2 or 3 choices for the states of the lights we have $m > 1$ choices for which m is not divisible by 5, there is a solution for every initial configuration to the target configuration. This has to do with the matrix A, which has an inverse when m is not divisible by 5.

References

1. Pelletier, Don. Merlin's Magic Square. Mathematical Monthly, vol. 94, No. 2, 143-150. doi: 10.2307/2322414