The Fourth Dimension

1 Coordinate Systems

1. What dimension is the space consisting of all points on the surface of the chalkboard? Why?
2. What dimension is the inside of a circle? The boundary of a circle? Why?
3. What dimension is the space of all colored pixels on a standard RGB computer screen?

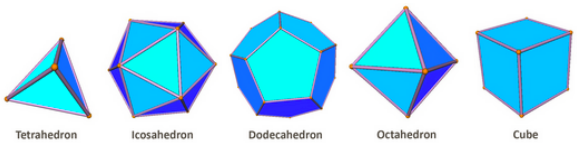
2 Distances

4. What is the distance between the points $(2019, 2020)$ and $(2023, 2025)$ on the plane? What is the distance between $(-1, 2)$ and $(3, 5)$? In general, how do we compute distance in the plane?
5. What is the distance from the origin $(0, 0, 0)$ to the point $(1, 2, 3)$ in three dimensional space? What is the distance between the points $(2, 5, 11)$ and $(3, 7, 13)$?
6. What is the distance between the points $(1, -1, 0, -1)$ and $(0, 1, 4, 9)$ in four dimensional space? Find two points in four dimensional space with integer coordinates that are exactly seven units apart.

3 Polyhedra

7. A polygon is 2 dimensional figure with straight sides. Here are some regular ones:

A polyhedron is a 3-dimensional figure with flat faces that are polygons. Here are some regular ones:



Fill in the blank: A 4-dimensional polytope is a 4-dimensional figure with ....

Are there any 1-dimensional analogs of polygons? What would be the regular ones?

8. Which 3-d polyhedron is most like a 2-d square? Suppose you were trying to describe this polyhedron to a 2-dimensional creature who couldn't imagine 3-dimensions. How could you describe it?

9. What is the 4-d analog of a cube? How many vertices (V), edges (E), faces (F), and hyperfaces (H) does it have? It may be helpful to think about how you could generate a cube beginning with a square, and then reason by analogy.

10. How would you draw a picture of a cube to show to a 2-dimensional creature? Can you draw a picture of a hypercube?

11. Which 3-d polyhedron is most like a 2-d triangle? How would you describe it to a 2-dimensional creature?

12. What is the 4-d analog of a tetrahedron? How many vertices (V), edges (E), faces (F), and hyperfaces (H) does it have? It may be helpful to think about generating a tetrahedron from a triangle first, and then reason by analogy.

13. For polyhedra, Euler's formula says that $V - E - F = 2$. Is there an analogous Euler's formula for 3-d polytopes? What should it say?

14. Can you generalize the square and the triangle to 5 dimensions?

15. Build a model of a 4-dimensional cube using Zome tools. You will actually be building the projection, or "shadow", of its edges in 3-dimensions.

16. How could you unknot a trefoil knot in 4-dimensions?

17. One morning, I woke up and got dressed. When I went to put on my shoes, both shoes were left shoes. I immediately suspected that a 4-dimensional creature had gotten in my house and played a trick on me. How did he get in my house and how did he turn my right shoe into a left one?

18. Is it possible for an ant to begin at one vertex of a cube, crawl along all the edges, and then return to its starting point without retracing any portion of its path? Is this possible on a hypercube? (Make sure that you have an accurate hypercube sketch first.)

19. A ball of radius r consists of the set of all points whose distance from a given point (the center of the ball) is r or less. Describe how a ball looks in one, two, or three dimensions. Then state a formula for the length, area, or volume of such a ball. Based on these expressions, what sort of formula would you expect for the volume of a four-dimensional ball of radius r?