CHMC Advanced Group:
Nim

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1 Introduction

In mathematics, the study of certain types of games has been a fruitful pursuit leading to a deeper understanding of rich structures. One such area is the study of combinatorial games, and a classic example of this type of game is Nim. The study of Nim helped to drive a lot of preliminary research in combinatorial games and many results relate more complex games back to Nim as analysis can more easily be completed in this setting. In this worksheet, we will explore the game of Nim and some of the strategies for winning under different types of game play.

2 Nim

In this section, we introduce the game of Nim and familiarize ourselves with the game play for various numbers of piles.

The game of Nim is a two-player game. In this game, there are $n \geq 2$ piles of stones, with pile i consisting of $m_i \geq 0$ stones. During each turn, a player will select a non-empty pile and remove as many stones as they wish from it. The game ends when there are no stones remaining in any of the piles.

There are two versions of the game, *Normal Play* and *Misere Play*. In the first version, the player to make the last move wins; in the second version, the player to make the last move loses.

Exercise 2.1 Consider a game of Nim involving two piles of stones, both of which have 10 stones to start. Play a few games with a partner under the Normal Play rules. Can you determine a strategy for player one or player two to follow that guarantees a win? Do the same with Misere Play rules.

Exercise 2.2 Instead of starting with 10 stones each, suppose the first pile has 11 stones. Play a few games using Normal Play rules and a few games using Misere Play rules. Are there strategies that guarantee one of the players a win under either set of rules?

Exercise 2.3 Play Nim with three piles, each of which have 10 stones, using Normal Play rules. Then, play Nim with three piles two of which have 11 stones and the third with 10 stones. Finally, try starting with the first pile having 12 stones, the second having 11 stones and the third with 10 stones. Can you develop a strategy for either player in any of the three setups?

3 Nim Positions

For a two-player game, we classify each position as a P-position or an N-position as follows:

A game is in a P-position if it secures a win for the Previous player (the one who just moved).

A game is in a N-position if it secures a win for the Next player.

In Normal Play Nim using two piles, (0,1) is in N-position while (1,1) is in P-position. In Misere Play Nim using two piles, (0,1) is in P-position while (1,1) is in N-position. A **terminal position** is a position from which there are no more moves.

The following labelling procedure allows us to map out a game of Nim under Normal Play.

1. For each terminal position, label this as a P-position.

2. For each position that has a move to a P-position, label it an N-position.

3. For any position that can only move to an N-position, label it a P-position.

4. If any positions remain unlabelled, go back to step 2 and repeat until all positions are labelled.

For Misere Play, in step 1, label all the terminal positions as N-positions and follow steps 2 - 4 as above.

Exercise 3.1 In Nim involving two piles under Normal Play, determine whether each of the following positions is a P-position or an N-position: (0,0), (0,1), (0,n) where $n \geq 1$, (1,1), (1,2), (2,2), (3,5). Do you see a pattern?

Exercise 3.2 Do the same analysis as in the previous exercise, for the three pile Nim under Normal Play with starting configuration (2,3,4) (you will need to determine the labelling for positions starting at (0,0,0) and moving backwards until you hit (2,3,4)). Do you see any patterns with which positions are P-positions and which are N-positions?

Exercise 3.3 Repeat the previous exercises under Misere Play rules.

Exercise 3.4 Do you notice any similarities in the position labels for corresponding positions under Normal Play and Misere Play in two pile Nim? How about in three pile Nim?

4 Binary operations and Nim-sums

In this section, we learn how to represent non-negative integers in binary and operations on these binary representations.

The binary representation of a non-negative integer is the unique string bits, represented by 1's and 0's, expressing the integer as a unique sum of positive integral powers of 2. For instance,

$$1 = 1\cdot 2^0 \quad 2 = 1\cdot 2^1 + 0\cdot 2^0 \quad 3 = 1\cdot 2^1 + 1\cdot 2^0 \quad 4 = 1\cdot 2^2 + 0\cdot 2^1 + 0\cdot 2^0 \quad 6 = 1\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0$$

The corresponding binary representations are $1 = 1_2, 2 = 10_2, 3 = 11_2, 4 = 100_2, 6 = 110_2$. In the representation of 6 in binary, the first bit is a 0, the second bit is a 1 and the third bit is also a 1.

Exercise 4.1 Determine the binary representations of the integers from 10 to 20. Also, determine what integers are represented by $11_2, 111_2, 1111_2, 11111_2$ in binary. Also do this for the integers with binary representations are $100_2, 1000_2, 10000_2$.

The XOR operation $\oplus$ for two bits is defined as $0_2 \oplus 0_2 = 0_2 = 1_2 \oplus 1_2$ and $1_2 \oplus 0_2 = 1_2 = 0_2 \oplus 1_2$. This can be expanded to binary representations by applying the operation bit by bit.

For instance, $101_2 \oplus 110_2 = 010_2$ since $1_2 \oplus 1_2 = 0_2$, $0_2 \oplus 1_2 = 1_2$ and $1_2 \oplus 0_2 = 1_2$.

Exercise 4.2 Calculate $110_2 \oplus 101_2$, $111001_2 \oplus 001101_2$, $100101_2 \oplus 111001_2$.

In the next exercise, you will show that on a bit basis, $\oplus$ is associative, that is $(a\oplus b)\oplus c = a \oplus (b \oplus c)$ where a, b, c can be 0 or 1. What this allows use to do is write $a \oplus b \oplus c$ since it does not matter the order in which we evaluate the XOR operations.

Exercise 4.3 Show that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ where a, b, c can be 0 or 1. For instance, $(0 \oplus 1) \oplus 0 = 1 \oplus 0 = 1$ and $0 \oplus (1 \oplus 0) = 0 \oplus 1 = 1$. I have suppressed the subscript 2 for ease of writing here, but consider everything here in binary representation.

In the following exercise, you will show that on a bit basis, $\oplus$ is commutative, that is $a \oplus b = b \oplus a$ where a, b can be 0 or 1. Combined with the above observation that $\oplus$ is associative, one can write $x_1 \oplus \cdots \oplus x_k$ in any ordering of $1, 2, \ldots, k$, without changing the value of the expression. This makes calculations simpler at times.

Exercise 4.4 Show that $a \oplus b = b \oplus a$ where a, b can be 0 or 1.

Define the Nim-sum of a position in a game of Nim of n piles as the value $x_1 \oplus \cdots \oplus x_n$ where x_i denotes the number of stones in pile i. For instance, the Nim-sum of the position $(2, 3, 5)$ is $2 \oplus 3 \oplus 5 = 010_2 \oplus 011_2 \oplus 101_2 = 100_2$.

Exercise 4.5 In Nim involving two piles, determine the Nim-sum of each of the following positions: (0,0), (0,1), (0,n) where $n \geq 1$, (1,1), (1,2), (2,2), (3,5). Compare these values to the N-positions and P-positions under Normal Play rules. Do you see a pattern?

Exercise 4.6 Do the same analysis as in the previous exercise, for the three pile Nim under Normal Play with starting configuration (2,3,4) (determine the Nim-sums for positions starting at (a, b, c) where $a, b, c \geq 0$, $a \leq 2, b \leq 3, c \leq 4$). Do you see any patterns with which positions are P-positions and which are N-positions, which you previously determined, and the values of Nim-sums of these positions?

Exercise 4.7 Repeat the previous two exercises under Misere Play rules.

5 Determining a Winning Strategy

There are two important pieces of information that lead one to develop a winning strategy for Nim, under either Normal Play or Misere Play.

The first is that is a position has Nim-sum equal to zero, any move from that position will result in a non-zero Nim-sum.

The second is that in a position with a non-zero Nim-sum, there is always a move to a new position that will have a Nim-sum equal to zero.

Exercise 5.1 Using the two facts stated above and connection between the Nim-sum of positions and the classification of N-positions and P-positions, try to develop a winning strategy for the first player in a two pile game of Nim under Normal Play rules where the piles start out uneven.

Exercise 5.2 Repeat the previous exercises, but start the piles with equal numbers of stones. Is there a winning strategy for the first player or the second player?

Exercise 5.3 Try the same analysis that you did in the previous two exercises using Misere Play rules.