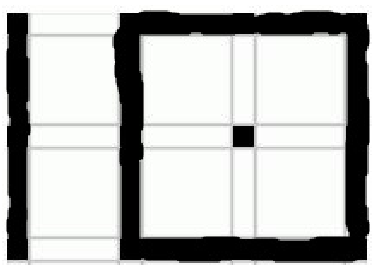
Math Circle Worksheet
Dots and boxes, pt. 2

11/17/18

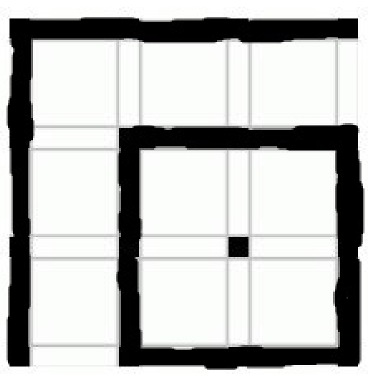
1 Winning from a set configuration

In the following configurations, it is your turn to go next. Assuming your opponent plays optimally (the best they possibly can), how might you win?

Exercise 1.1 Place an edge to ensure that, no matter how your opponent plays, you will win. In this scenario, would you be player A or player B?

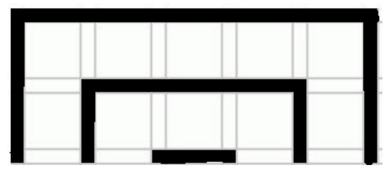


Exercise 1.2 Place an edge to ensure that, no matter how your opponent plays, you will win. In this scenario, would you be player A or player B?

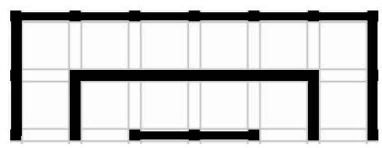


Exercise 1.3 Place an edge to ensure that, no matter how your opponent plays, you will win. In this scenario, would you be player A or player B?

Exercise 1.4 Can you place an edge to ensure that, if your opponent plays to capture as many squares as possible in each turn, you will win? In this scenario, would you be player A or player B? What possible plays are there?

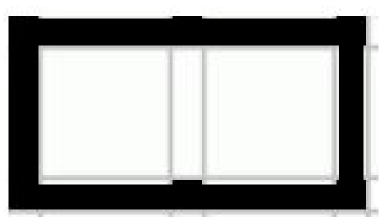


Exercise 1.5 Can you place an edge to ensure that, if your opponent plays to capture as many squares as possible in each turn, you will win? In this scenario, would you be player A or player B? What possible plays are there?



Exercise 1.6 Suppose that, in the above two exercises, your opponent does not necessarily play "greedily." Can your moves be countered?

You may have noticed, in the last two exercises, that it's beneficial to give up some squares you could easily take to ensure that you can pick up a longer chain. In fact we call two squares, right next to each other, that are missing a single edge, a **doublecross**. Here is an example of a doublecross configuration:



If your opponent takes a doublecross, then they gain two squares but must also make the next move. If that next move sets up for you to sweep through a "tunnel" of squares, then you can easily pick up more boxes than you gave up.

2 Long chains

A **long chain** is a string of 3 or more consecutive squares, each sharing an empty edge, where each square has exactly two missing edges. Another way of thinking about long chains is that they are “tunnels” snaking through the dotted grid. For example, in the configurations from exercises 1.4 and 1.5 there are two long chains each, whereas the configuration in exercise 1.3 only has one long chain. In exercise 1.5, the “top” long chain has length 8, and the “bottom” long chain has length 4.

Exercise 2.1 Why doesn’t the configuration in exercise 1.3 have two long chains?

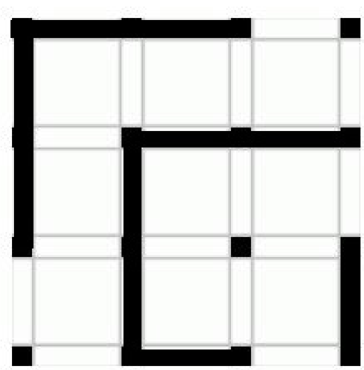
Counting long chains turns out to be one strategy to win. It can be countered by a wary opponent, but it sets the stage for some of the more advanced dots-and-boxes strategies. Thus, the rest of this worksheet will be focused on long chains.

It’s worth noting that, in this framework, we don’t count “closed loops” as long chains. An example of this is given in exercise 1.2: there is only one long chain, the sequence of five squares on the left- and top-edges of the 2×2 square in the bottom right.

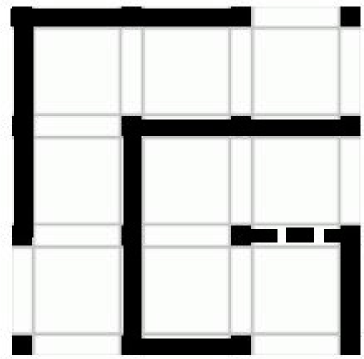
Exercise 2.2 The configuration in exercise 1.1 has no long chain; why?

Exercise 2.3 In the following configuration it’s your turn: where could you place an edge to ensure that you can claim the long chain on top? (This is the same configuration from exercise 1.3.)

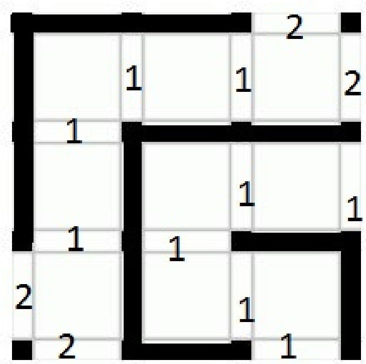
The following example is taken from Erlekamp’s book “The Dots and Boxes Game: Sophisticated Child’s Play”. Consider the configuration below, in which it is your turn to play. You are player A (why?).



There are two chains in the work (the top-left chain of length 3, and portions of a chain in the bottom-right). You would like to take both chains; how might you do it? Place an edge (dashed) like so:

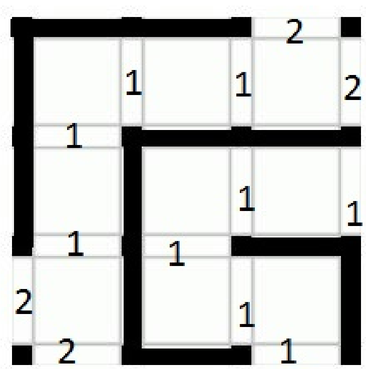


This turns the bottom-right of the grid into a length 4 long chain, but also ensures that you can claim both chains; on top of that, you don't need to give any doublecrosses to your opponent. Let's label the remaining edges to make our discussion easier:



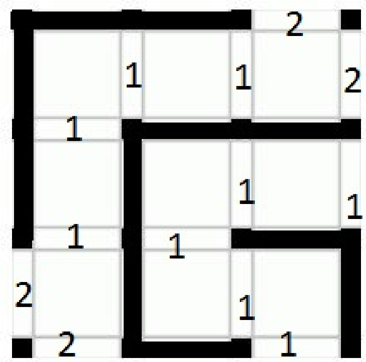
If your opponent places an edge at any spot labeled "1," you can claim the rest of the squares in that chain and then play an edge at a spot labeled "2". If your opponent then plays at another spot labeled "1", you can claim the rest of that chain as well as the square next to the "2" you had previously claimed. This then leaves one square for the opponent to claim.

Exercise 2.4 Try a game like the above! In the following game board, have the next move take a "1" edge, the following move take a "2" edge, and the next move take a "1" edge. What happens next?



Exercise 2.5 Suppose that the sequence of moves for the board is instead: “1” edge, “2” edge, “2” edge. What are the possibilities for taking that last “2” edge? How does each possibility change the game? Can you still win?

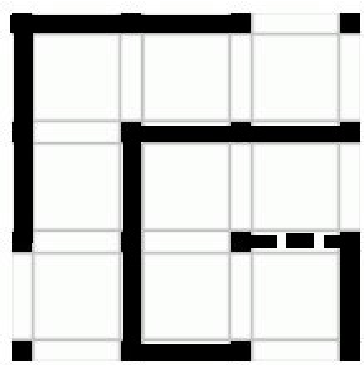
Now let’s go back to this configuration:



Suppose your opponent picks a “2” edge instead of a “1” edge.

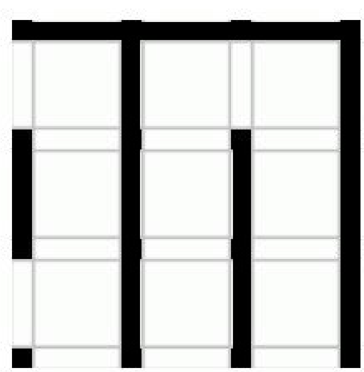
Exercise 2.6 To ensure a win, you want to pick a “2” edge that is far from the “2” edge your opponent picked; why?

In this way, we can see that by making this original dashed move the player can claim two long chains:

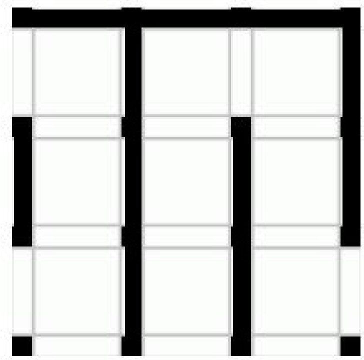


The following exercises, similar to the above, are again taken from Erlekamp’s book.

Exercise 2.7 Where should you place a single edge to ensure you can claim two long chains? Why are you guaranteed two long chains?



Exercise 2.8 Where should you place a single edge to ensure you can claim one long chain? Why are you guaranteed a long chain?



Exercise 2.9 How are the previous two exercises related? In the configuration for exercise 2.8, is it player A or player B's turn? What about exercise 2.9?

3 A winning strategy

Suppose that players A and B are playing on a rectangular grid of m dots by n dots. Then player A has a winning strategy: if m and n are both odd, then player A should play to keep the number of long chains odd; if either m or n are even, then player A should play to keep the number of long chains even.

We don't have enough time in this session to prove that this is, in fact, a winning strategy, but the website `https://www.math.ucla.edu/~tom/Games/dots&boxes.html` gives an outline of the proof[1]. Note that this website also lets you pick a grid size for dots-and-boxes, and has you play against a computer. See if you can beat them!

[1] You can also google "dots and boxes" and follow the link that takes you to the UCLA website.