Math Circle Worksheet
The Game of Hex, pt. 2

10/20/18

1 Can Hex end in a tie?

Hopefully in playing Hex, you noticed that ties never occurred. This turns out to be true in general: Hex can never end in a tie. Below are two proofs of this fact. The first is intended to be more intuitive and less rigorous, whereas the last proof presented is rigorous and uses some ideas from graph theory.

1.1 Intuitive proof

Play a game of Hex against an opponent, except this time you colour your hexagons red, and they cut out their hexagons (including the boundary of the hexagon). Suppose each of you play until every hexagon in the grid is either coloured red or cut out. Hold the paper up by the two red edges, and pull your hands apart. One of two things will happen:

- your hands will move apart, or
- the paper will keep your hands from moving apart.

Exercise 1.1 Suppose your hands move apart; why does this happen? How would you interpret this in terms of the Hex game you just played?

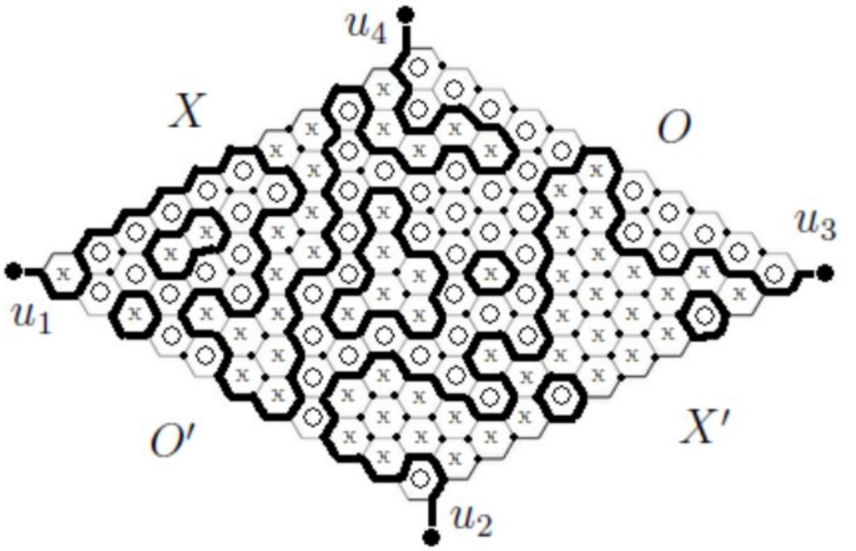
Exercise 1.2 Suppose your hands are kept from moving apart. Again, why does this happen? How would you interpret this in terms of the Hex game you just played?

Exercise 1.3 Using the physical interpretation of a Hex game above, why is a tie impossible?

1.2 Rigorous proof

The set up for the rigorous proof is as follows: suppose you play a game of Hex and fill in all hexagons. Whenever an 'X' hexagon shares an edge with an 'O' hexagon, color that edge black; we also include an edge where the 'X' edges of the Hex grid meet the 'O' edges. See the figure on the following page; 'X' and 'X' ' are the two opposite 'X' edges, and u_1, u_2, u_3, u_4

are the edges corresponding to the borders of the ‘X’ and ‘O’ edges. The rest of this proof involves showing that there must be a path of edges connecting two corners (where an ‘X’ edge meets an ‘O’ edge) of the grid.



Start at any edge, say u_1. We proceed along the path of edges that follow from u_1, which corresponds to a string of ‘X’s and ‘O’s.

Exercise 1.4 Why are we guaranteed that u_1 will connect to a path of edges that goes into the Hex board? Hint: what are the possible markers that can be placed on the hexagon closest to u_1?

Exercise 1.5 As we follow this path of edges, we’ll never end up at a “dead-end” of the path anywhere in the grid; why? What would it take for the path to end at a dead-end?

To finish the proof, we need the following lemma. This can be treated as a challenge problem; if you have time, try to prove it. Otherwise, use it as a black box in the rest of the proof.

Exercise 1.6 Extra problem: Prove that for a finite graph in which every vertex is connected to at most two edges, the graph must either be composed of

1. isolated vertices,
2. a simple cycle (loop of edges), or
3. a simple path (a string of edges).

By the above lemma, the path of edges we’re following must be a simple path.

Exercise 1.7 Prove this last statement. Why can’t our path be a cycle or a collection of isolated vertices?

Exercise 1.8 Can our path return to u_1? Since it must be a simple path, where must the path end?

Exercise 1.9 Notice that when we have such a path of edges, there must be 'X's on one side of the path and 'O's on the other. Why does this guarantee that one of the two players has a path connecting opposite corners?

Hence, whenever a Hex board is completely filled in, there must be a collection of 'X's or a collection of 'O's that join opposite edges of the grid.

2 Winning strategies

A winning strategy is defined to be a sequence of moves, dependent on your opponents moves, that ensures you win. Winning strategies for Hex exist, for any grid size; we will prove this shortly. In practice, however, it is unknown what the actual winning strategy is for $k \times k$ grids, $k \geq 10$.

Thus, we will show that a winning strategy for player 1 always exists. Suppose, instead, that a winning strategy for player 2 exists. On player 1s first move, they should make a random move. In this situation, player 2 can be considered as the first player, and player 1 becomes, effectively, the second player. By assumption the second player has a winning strategy, and so player 1 does indeed have a winning strategy.

Exercise 2.1 This proof relies on the fact that a random marker on the board won't affect player 1's optimal strategy; why is this true? Hint: look at two cases, when the random move is part of the winning strategy, and when it isn't.

3 Why hexagons and not squares?

Hex is played with a hexagonal grid, and we saw above that a game cannot end in a tie; what happens if we play with a square grid instead?

For the square version of Hex, the grid is now a $k \times k$ board of squares. As before, two opposite edges are marked 'X', and the other two edges are marked 'O', and players take turns marking squares. Recall that in the original version, two hexagons are connected if they have the same marker on them, and they share an edge. In the square version, two squares are connected if they share an edge. *If two squares only meet at a corner, they are not connected.* In this version, it is possible to end in a tie.

Exercise 3.1 Show that ties are possible in the square version of Hex. Hint: try to come up with tie games for a 4×4 board, and generalize the construction to arbitrary $k \times k$ boards.

Exercise 3.2 How do the grids of Hex and square-Hex relate? Hint: draw the Hex and square-Hex boards with a new representation: hexagons (or squares) are points, and two points are connected by a line if the corresponding hexagons (or squares) share an edge.

Exercise 3.3 Why do you think ties are impossible with a Hexagonal board, but possible with a square board?

Exercise 3.4 What happens if we allow two squares to be connected if they share a corner? How does the game change? Are ties possible now?