CHMC Advanced Group: Generating Functions

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1 Introduction

In mathematics, certain sequences of numbers are of interest for the properties the sequences possess. Sometimes, it is advantageous to encode these sequences as the coefficients of power series, that is, for a sequence $\{a_n\}_{n=0}^{\infty}$ there is an associated power series $\sum_{n=0}^{\infty} a_n x^n$. If one wants to explicitly determine the elements of a sequence given certain properties of the sequence, one profitable approach is to represent the sequence as a power series, an example of a generating function. Then, using properties of such series and the original sequence, one can explicitly determine the elements of the sequence $\{a_n\}_{n=0}^{\infty}$. In this worksheet, we will see this approach in action with three different types of problems.

2 Partitions

Let n be a positive integer, and consider a pile of n pencils. We may wish to separate the pencils into non-empty groups of pencils, such that the first group has at least as many pencils as the second group, continuing in this manner until all the pencils have been accounted for. The sizes of these groups constitute a **partition** of the integer n. For example, when $n = 5$, some possible partitions are $3, 2$, and $4, 1$, and $3, 1, 1$. An interesting question one may ask is, "How many partitions are there for a given positive integer n?"

Exercise 2.1 Determine the number of partitions for $n = 1$, $n = 2$, and $n = 3$. You may try to create a systematic approach, say by looking at how big the starting group is.

Exercise 2.2 Determine the number of partitions for $n = 4$, $n = 5$, and $n = 6$. Again, try using a systematic approach to this.

Now, consider the monomial x^n and the possible ways of multiplying monomials of the form x^{k_i} such that $k_1 \geq k_2 \geq \cdots \geq k_j \geq 1$ and $x^{k_1} \cdot x^{k_2} \cdot \cdots \cdot x^{k_j} = x^n$.

Exercise 2.3 How does this problem relate to the problem of partitions? What if we vary j from $j = 1$ to $j = n$?

There is a way of connecting these two ideas into something called a **generating function**. In particular, if we denote by a_n the number of partitions of the positive integer n, we hope to write a function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ in some meaningful way.

We take a brief aside for a moment to discuss the notation used above. By $\sum_{n=1}^{\infty} a_n x^n$, I mean the sum of terms $a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. One can ask for given values of x whether the sum **converges** to some real number, that is, equals a real number. If not, we say the infinite sum diverges. When we talk about a formal power series, we do not necessarily concern ourselves with questions of convergence or divergence, but instead talk about properties of the coefficients of the power series and how we may use these properties to determine the coefficients.

Consider as before, writing the partitions for a given integer n in terms of the starting group size. Say, for $n = 5$, we look at the partition $2, 2, 1$. The first group size is given by 2, the second by 2, and the third by 1. We can write this equivalently as $x^2 \cdot x^2 \cdot x = x^5$. Convince yourself that this correspondence between partitions and multiplying monomials is unique in the sense that to each partition corresponds such a product and to every product corresponds a unique partition.

Exercise 2.4 Fix $n = 4$. Consider the product of $(1+x+x^2+x^3+x^4)(1+x^2+x^4)(1+x^3)(1+x^4)$ and count the coefficient in front of the x^4 term. Compare this to the value you calculated for the number of partitions of $n = 4$. Try to relate the monomial products in this multiplication to the partitions. Do the same for $n = 3$ with $(1 + x + x^2 + x^3)(1 + x^2)(1 + x^3)$ with the x^3 term. What do you observe?

Exercise 2.5 For $n = 5$ and $n = 6$, what product of polynomials and specific term do you suggest examining to determine the number of partitions of size $n = 5$ and $n = 6$

respectively? Try your guess and see if it matches the values you previously determined. Now, also check the coefficients of x, x^2, x^3, x^4 and determine how these relate to the values of the number of partitions you previously calculated.

Exercise 2.6 How do you think the generating function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ can be represented as a product of infinite sums of monomials (a possibly infinite product)? Try to justify your answer.

Exercise 2.7 For positive k, it is known that $1 + x^k + x^{2k} + \cdots = \frac{1}{1-x^k}$ when restricting to $-1 < x < 1$. How can you write your answer to the previous exercise in a more compact form using this equality?

Now, consider partitions where only an odd size groups are allowed. For example, with $n = 3$ we such partitions are 3 and $1, 1, 1$ and for $n = 4$ such partitions are $3, 1$ and $1, 1, 1, 1$.

Exercise 2.8 Determine the number of partitions consisting of only odd sized groups for $n = 2, 3, 4, 5, 6$.

Now, try to relate this to the product of polynomials.

Exercise 2.9 Consider the following product $(1+x+x^2+x^3+x^4+x^5+x^6)(1+x^3+x^6)(1+x^5)$ and relate the coefficients to the number of partitions consisting only of odd groups that you previously calculated.

Exercise 2.10 For the generating function $f(x) = \sum_{n=1}^{\infty} o_n x^n$ where o_n counts the number of partitions solely consisting of odd group sizes, try to determine what product of infinite sums results in the generating function.

Try to adapt your strategy to determine the number of partitions where the group sizes are distinct. For example, for $n = 6$ this consists of 6, and $5, 1$, and $4, 2$, and $3, 2, 1$.

Exercise 2.11 First, determine the number of such partitions d_n for $n = 3, 4, 5, 6, 7$. Then, relate these partitions to product of monomials. Finally, relate this to a product of polynomials as before. Then, generalize to the infinite case.

3 Fibonacci Numbers

Now, we consider a famous sequence of numbers. It is given by a **recursion**, an equation involving previous terms in defining subsequent terms.

The equation is as follows $a_0 = 0, a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$.

Exercise 3.1 Determine $a_0, a_1, \ldots, a_{10}$ using the equation given above.

Exercise 3.2 Show that $a_n \leq 2^n$ for $n \geq 0$. This will require using induction and applying the assumption to the defining recursive equation.

Now, suppose that some generating function exists, with $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We want to determine a closed form for the coefficients a_n.

Exercise 3.3 For each $n \geq 1$, multiply $a_{n+1} = a_n + a_{n-1}$ by x^n and add all these equations together starting at $n = 1$. Write down the left hand side in terms of $f(x)$ and x. You should get a rational function for $\sum_{n=1}^{\infty} a_{n+1} x^n$. Now, write down the right hand side in terms of $f(x)$ and x, that is, $\sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$.

Exercise 3.4 With the equation you determined above, solve for $f(x)$ in terms of x. You should get a rational function with a linear numerator and quadratic denominator.

Exercise 3.5 With the quadratic denominator, factor it into the form $(1 - r_+ x)(1 - r_- x)$. Determine the values of r_- and r_+.

Exercise 3.6 Show that if $x^2 + bx + c = (1 - sx)(1 - tx)$, then

$$\frac{x}{x^2 + bx + x} = \frac{1}{s - t}\left(\frac{1}{1 - sx} - \frac{1}{1 - tx}\right)$$

Exercise 3.7 Show that for an infinite series defined by $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$, this can be written as $\frac{1}{1-\frac{1}{2}} = 2$. To do this, suppose $S(\frac{1}{2}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$. What is $\frac{1}{2} \cdot S(\frac{1}{2})$? Now, compute $S(\frac{1}{2})(1 - \frac{1}{2}) = S(\frac{1}{2}) - \frac{1}{2} \cdot S(\frac{1}{2})$. From this, solve for $S(\frac{1}{2})$ in terms of a real number value.

We repeat the exact same process below, but this time for arbitrary r. Again, we do not concern ourselves with convergence but instead look at this as a formal power series.

Exercise 3.8 Show that for an infinite series defined by $\sum_{n=0}^{\infty} r^n$ where $r \neq 1$, this can be written as $\frac{1}{1-r}$. To do this, suppose $S(r) = \sum_{n=0}^{\infty} r^n$. What is $rS(r)$? Now, compute $S(r)(1 - r) = S(r) - rS(r)$. From this, solve for $S(r)$ in terms of r.

Exercise 3.9 Apply the formula from the previous exercise to the terms $\frac{1}{1-xr_-}$ and $\frac{1}{1-xr_+}$ to get series of the from $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ respectively. Now, apply exercise 3.5 and 3.6 to simplify the expression from exercise 3.4. Combine all the series to get an equation $f(x) = \sum_{n=0}^{\infty} a_n x^n$. This determines a formula for a_n for $n \geq 0$ that is not recursive.

4 Catalan Numbers

Consider for positive integer n a group of $2n$ pegs spaced evenly around a circle. For such an arrangement, consider tying a string to a pair of pegs. Keep doing this until all pegs are accounted for. We call a diagram valid if every peg has a string tied to it, no peg has more than one string tied to it, and no two strings ever cross.

Consider for example $n = 2$ and pegs placed at 3, 6, 9, and 12 on a clock. Valid arrangements are $(3,6)(9,12)$ and $(3,12)(6,9)$ since every peg has exactly one string and no strings ever cross. An invalid arrangement is $(3,9)(6,12)$ since the strings cross.

We want to determine c_n, the number of valid arrangements of strings on $2n$ pegs. By convention, $c_0 = 1$ and $\binom{0}{0} = 1$.

Exercise 4.1 Calculate c_n for $n = 0, 1, 2, 3, 4$. It is highly encouraged that you draw out diagrams to help keep track of all the possible valid arrangements.

Now, suppose I know the values to $c_1, c_2, \ldots, c_n$ and I want to determine c_{n+1}. We will try to relate c_{n+1} to the values of $c_1, \ldots, c_n$. Before tackling this in the abstract, consider doing this with low values of n.

Exercise 4.2 Try to determine the value of c_3 using the value of c_1 and of c_2. Since strings connect a pair of pegs, consider which pegs of 2,3,4,5,6 can connect to peg 1 that will make a valid arrangement. Once you have determined that, draw a diagram for each possible peg that can connect with peg 1. Consider the pegs to the left and right of the string tied to peg 1. How many pegs are there on the left and on the right? Now, how many arrangements of strings can be made on the group of pegs to the left and the group of pegs to the right? How do these relate to c_0, c_1, and c_2. Furthermore, choosing an arrangement to the left does not affect the choice of an arrangement to the right. Hence, we can multiply the number on the left with the number on the right to determine the total number of arrangements for a given peg 1 choice. We can add up all the products to determine c_3.

Exercise 4.3 Count by hand and use the above process to determine c_4. Check that they match.

Exercise 4.4 Determine for general positive integer n, the value of c_n in terms of the values of $c_0, c_1, c_2, \ldots, c_{n-1}$.

Exercise 4.5 Suppose the generating function of Catalan numbers is given by $C(x) = \sum_{n=0}^{\infty} c_n x^n$. Consider $C(x)^2$ and group the coefficients according the each power of x (you only need to do this for low powers of x to see the pattern). What pattern do you see? Is there a power of x you can multiply $C(x)^2$ by so that this looks similar to $C(x)$? If so, what is it? What do you need to add to recover $C(x)$?

This **recurrence** of the coefficients, determines a relation where $C(x)$ is defined in terms of itself.

Exercise 4.6 Use the quadratic equation to solve for $C(x)$. You will get two solutions, but for reasons beyond this worksheet we will take the one where the square root is subtracted.

For the next step in determining the coefficients, we use the identity

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n$$

Exercise 4.7 Substitute $-4x$ in for y in the above identity. Then, multiply by -1 and add 1. Finally, divide by $2x$. Next, index so that the index starts at $n = 0$.

Exercise 4.8 Show that

$$\frac{1}{2} \frac{1}{2n+1} \binom{2n+2}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$

Using this and the series you computed in the previous exercise, what is the value of c_n for $n \geq 0$?