CHMC Advanced Group: Metric spaces

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1 Introduction

A "metric" is a notion of distance; in math, metrics provide a way to compare objects: the distance between two numbers, the distance between two points in a plane, etc. We can take the properties we like about the notion of distance in the plane, though, and generalize to other kinds of spaces. For example, one can define a nice notion of distance for sequences of 0s and 1s. One can then generalize this metric to strings of letters, and onto words.

This worksheet will explore each of the metrics mentioned above.

2 The Euclidean and Taxicab metrics

In the plane $\mathbb{R}^2 = \{(x, y) : x, y \text{ are real numbers}\}$ we have the usual distance formula: for two points $(x_1, y_1), (x_2, y_2)$, we define their distance d_E by the formula

$$d_E((x_1, y_1), (x_2, y_2)) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

This is the standard **Euclidean metric**. Later in this worksheet we will define the notion of a metric more precisely, but for now it suffices to treat the names "metric" and "distance function" interchangeably.

As an example of computing the distance between two points, let's find the distance between $(1, 2)$ and $(3, 4)$: we compute

$$\begin{aligned} d_E((1,2),(3,4)) &= \sqrt{(1-3)^2 + (2-4)^2} \\ &= \sqrt{(-2)^2 + (-2)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8}. \end{aligned}$$

Let's establish a few basic properties of d_E.

Exercise 2.1 Show that $d_E((x_1, y_1), (x_2, y_2)) \geq 0$ for all points $(x_1, y_1), (x_2, y_2)$.

Exercise 2.2 Show that $d_E((x_1, y_1), (x_2, y_2)) = 0$ if and only if $(x_1, y_1) = (x_2, y_2)$.

Exercise 2.3 Show that $d_E((x_1, y_1), (x_2, y_2)) = d_E((x_2, y_2), (x_1, y_1))$.

Another property that d_E satisfies, which we won't prove here, is called the **triangle inequality**:

$$d_E(x, z) \leq d_E(x, y) + d_E(y, z),$$

where x, y, z are any points in the plane.

Exercise 2.4 Verify that $d_E((1, 2), (3, 4)) \leq d_E((1, 2), (0, 0)) + d_E((0, 0), (3, 4))$.

One concept that we'll explore throughout this worksheet is the notion of a disk. In the plane, we define the **closed disk of radius** r centered at the point (x_1, y_1), denoted $D((x_1, y_1), r)$, to be the set

$$\begin{aligned} D((x_1, y_1), r) &= \{(x, y) : d_E((x_1, y_1), (x, y)) \leq 1\} \\ &= \{(x, y) : \sqrt{(x_1 - x)^2 + (y_1 - y)^2} \leq 1\}. \end{aligned}$$

The **open disk of radius** r, denoted $D^\circ((x_1, y_1), r)$, is defined in the same way with a strict inequality instead:

$$\begin{aligned} D^\circ((x_1, y_1), r) &= \{(x, y) : d_E((x_1, y_1), (x, y)) < 1\} \\ &= \{(x, y) : \sqrt{(x_1 - x)^2 + (y_1 - y)^2} < 1\}. \end{aligned}$$

Sometimes we'll refer to the **unit closed (or open) disk**, which will be a disk with radius 1.

In the next few exercises, you should use the Euclidean metric d_E to measure distances.

Exercise 2.5 Is $(1, 0)$ in the closed unit disk $D((0, 0), 1)$? In the open unit disk $D^\circ((0, 0), 1)$?

Exercise 2.6 Is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ in the closed unit disk $D((0, 0), 1)$? In the open unit disk $D^\circ((0, 0), 1)$?

Exercise 2.7 Is $(\frac{1}{2}, \frac{1}{2})$ in the closed unit disk $D((0, 0), 1)$? In the open unit disk $D^\circ((0, 0), 1)$?

Exercise 2.8 Sketch the closed unit disk in the plane. (Recall that the equation $x^2 + y^2 = r^2$ defines a circle, centered at $(0, 0)$, with radius r.)

The metric d_E underlies much of Euclidean geometry, sometimes explicitly, usually implicitly. The next metric we'll explore, however, will cause familiar shapes to look quite different.

The **taxicab metric**, denoted d_t, is defined by the following equation: if (x_1, y_1) and (x_2, y_2) are two points in the plane, then their taxicab distance is

$$d_t((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Here, $|x|$ denotes the absolute value of the real number x.

For example, the taxicab distance between the points $(1,2)$ and $(3,4)$ is

$$\begin{aligned} d_t((1,2),(3,4)) &= |1-3| + |2-4| \\ &= |-2| + |-2| \\ &= 2+2 \\ &= 4. \end{aligned}$$

Notice that $d_E((1,2),(3,4)) = \sqrt{8} \approx 2.82843$, while $d_t((1,2),(3,4)) = 4$, so the points $(1,2)$ and $(3,4)$ have a larger taxicab distance than Euclidean distance.

The next few exercises are similar to Exercises 2.5–2.8. This time, however, you should use the taxicab metric d_t to measure distance. To be explicit, whenever you see $D((0,0),r)$, for example, you should work with the set

$$D((0,0),r) = \{(x,y) : d_t((x,y),(0,0)) \leq 1\}.$$

Exercise 2.9 Is $(1,0)$ in the closed unit disk $D((0,0),1)$? In the open unit disk $D^\circ((0,0),1)$?

Exercise 2.10 Is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ in the closed unit disk $D((0,0),1)$? In the open unit disk $D^\circ((0,0),1)$? How does this compare to exercise 2.2?

Exercise 2.11 Is $(\frac{1}{2}, \frac{1}{2})$ in the closed unit disk $D((0,0),1)$? In the open unit disk $D^\circ((0,0),1)$? How does this compare to exercise 2.3?

Exercise 2.12 Sketch the closed unit disk in the plane, with respect to taxicab distance. How does this closed disk compare to the unit disk from exercise 2.4?

The last few exercises for this section will establish some important properties we want from an arbitrary metric.

Exercise 2.13 Prove that $d_t(x,y) \geq 0$ for all points x, y in the plane. Note that here, we think of x as the pair (x_1, y_1), and y as the pair (x_2, y_2), etc.

Exercise 2.14 Prove that $d_t(x,y) = 0$ if and only if $x = y$.

Exercise 2.15 Prove that $d_t(x,y) = d_t(y,x)$.

Finally, unlike d_E, we will prove that d_t satisfies the triangle inequality. Recall that if a and b are any real numbers, then $|a-b| \leq |a| + |b|$.

Exercise 2.16 Prove that $d_t(x,y) \leq d_t(x,z) + d_t(z,y)$ for any points x, y, z in the plane.

3 The railway metric

Here we'll take a look at a different kind of metric on $\mathbb{R}^2$. The metric takes its name from, as you may have guessed, railways. The idea is that there is a central hub that trains can only travel in straight lines to or from a central hub. Thus, if a train wants to travel from the point $(1,0)$ to the point $(1,1)$, then it can't travel in a straight line upwards. Instead, it must travel to the central hub at $(0,0)$, and then it can travel from there to the point $(1,1)$. If we measure the Euclidean distance this train travels, we see that it is

$$d_E((1,0),(0,0)) + d_E((0,0),(1,1)) = 1 + \sqrt{1+1} = 1 + \sqrt{2}.$$

Even though the Euclidean distance between $(1,0)$ and $(1,1)$ is just 1, the train had to travel much further because it had to pass through the origin.

The **railway metric** d_r is defined by the exact same reasoning above: if (x_1,y_1) and (x_2,y_2) are two distinct points in the plane, then their railway distance is defined to be

$$d_r((x_1,y_1),(x_2,y_2)) = d_E((0,0),(x_1,y_1)) + d_E((0,0),(x_2,y_2)) = \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}.$$

If (x_1,y_1) and (x_2,y_2) are the same point, then the train does not have to move anywhere; it is already at its destination! Thus, we add the special case $d_r(x,x) = 0$ to the definition of d_r.

Exercise 3.1 What is $d_r((1,0),(-1,0))$? What about $d_r((1,0),(2,0))$?

Exercise 3.2 What is the open unit disk $D^\circ((0,0),1)$, with respect to d_r? Sketch the set.

Exercise 3.3 What is the open unit disk $D^\circ((1,0),1)$, with respect to d_r Sketch the set.

Exercise 3.4 Finally, what is the open unit disk centered at the point $(\frac{1}{2},0)$? Again, sketch it. Also sketch the closed disk centered at the origin, with radius $\frac{1}{2}$. What do you notice?

Exercise 3.5 Compare your sketches with those drawn in section 2. What do you notice?

This last optional exercise will be a variant of the railway metric, before we explore some different kinds of metrics.

Exercise 3.6 *What if we don't require the train to pass through the origin if its destination and origin lie on the same straight line passing through the origin? In other words, the train could travel directly from $(1,0)$ to $(2,0)$ without having to pass through $(0,0)$. In this situation, we define d_r by

$$d_r(x,y) = \begin{cases} d_E(x,y), & x = ay \text{ for some real number } a, \\ d_E(x,(0,0)) + d_E((0,0),y), & \text{otherwise.} \end{cases}$$

Here, saying $x = ay$ is equivalent to the requirement that x and y lie on the same line passing through the origin. Show that this new d_r is indeed a metric, and repeat exercises 3.1 through 3.5. How do your sketches change?

4 General metric spaces

A **metric space** is a set of objects X together with a function d, where d accepts a pair of elements of X and returns a non-negative number. For example, for the Euclidean metric d_E, the set X was 2-dimensional space, and the function was defined via the formula

$$d_E((x_1, y_1), (x_2, y_2)) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In this case, d_E always returns a non-negative number, but we saw that it satisfies a few other properties: $d_E(x, y) = 0$ if and only if $x = y$, $d_E(x, y) = d_E(y, x)$, and $d_E(x, y) \leq d_E(x, z) + d_E(z, y)$. We also saw that the metrics d_t and d_r also satisfied all of these properties.

With these examples in mind, we define a **metric** d on a set X to be a function d that accepts two elements of X, and outputs a real number. Moreover, d must satisfy the following four properties:

- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) \geq 0$ for all x, y in X;
- $d(x, y) = d_d(y, x)$;
- $d(x, y) \leq d(x, z) + d(z, y)$.

Here's another interesting metric: let X be any set, and define the **discrete metric** d_d by the formula

$$d_d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Notice that we can use *any* set X, and this function d_d will turn X into a metric space!

Exercise 4.1 Show that d_d satisfies the properties a metric is required to satisfy:

- $d_d(x, y) = 0$ if and only if $x = y$;
- $d_d(x, y) \geq 0$ for all x, y in X;
- $d_d(x, y) = d_d(y, x)$;
- $d_d(x, y) \leq d_d(x, z) + d_d(z, y)$.

Exercise 4.2 Consider $\mathbb{R}^2$ as a metric space with metric d_d defined above. What is the open unit disk centered at 0? What about the closed unit disk?

5 Metrics on sequences

In the rest of this worksheet, we will explore metrics on sequences. Recall that a **sequence** is an ordered collection of objects; we usually work with sequences of numbers, but sequences of letters also occur frequently in math. This section will explore a metric on sequences (first sequences of numbers, then sequences of letters) that are the same length. The following section will generalize this metric to sequences of letters that have different lengths.

To begin, let us define a family of sets to work with. The first kind of sequence we'll explore is the binary sequence, consisting of just zeros and ones, and we'll define the sets B_n by

$$B_n = \{(b_1, ..., b_n) : b_i = 0 \text{ or } 1 \text{ for each } i\}.$$

As an example, $B_1 = \{0, 1\}, B_2 = \{00, 01, 10, 11\}$, etc. In words, B_n consists of all possible sequences of length n consisting of just 0s and 1s[1].

Exercise 5.1 How many elements are in the set B_3? What are the elements?

Exercise 5.2 In general, how many elements are in the set B_n?

We can now define a metric. Let n be some fixed integer, and suppose $a = a_1a_2...a_n$ and $b = b_1b_2...b_n$ are two elements of B_n. We define the **Hamming distance** d_H between a and b to be the number

$$d_H(a, b) = |a_1 - b_1| + |a_2 - b_2| + \cdots |a_n - b_n| = \sum_{i=1}^{n} |a_i - b_i|.$$

For example, $d_H(00, 10) = |0 - 1| + |0 - 0| = 1 + 0 = 1$.

Exercise 5.3 What is the Hamming distance between 00101 and 11101?

Exercise 5.4 Verify that the Hamming distance d_H on B_n, for a fixed n, is in fact a metric. In particular, prove the following four propositions:

- $d_H(a, b) = 0$ if and only if $a = b$, i.e. a and b are the same sequence of 0s and 1s;
- $d_H(a, b) \geq 0$ for all sequences a and b;
- $d_H(a, b) = d_H(b, a)$;
- $d_H(a, b) \leq d_H(a, c) + d_H(c, b)$.

Exercise 5.5 In B_3, what are the elements of the open unit disk centered at 000, i.e. what are the elements of the set $D^\circ(000, 1)$? What are the elements of the closed unit disk $D(000, 1)$? How many elements are in each set?

Exercise 5.6 In B_n, how many elements are in the closed unit disk centered at $00 \cdots 0$ (the sequence consisting of n zeros)?

[1]Here we're identifying the sequence $(b_1, b_2, ..., b_n)$ with the binary number $b_1b_2...b_n$. For example, the sequence $(0, 1, 0, 0)$ is treated as the binary number 0100.

Suppose you want to cover the space B_n by closed unit disks: we define a **cover** of a set to be a family of sets X_i such that X is contained in the union of all the X_i. In other words, if x is in X, then x is in at least one of the X_i. One strategy is to just take every point in B_n, and take the closed unit disk around each point. This will work, but we may get a lot of overlapping sets.

Exercise 5.7 Show that the points $000, 001$, and 110, together with their closed unit disks, cover B_3. To be precise, show that if a is any sequence in B_3, then either $d_H(a, 000) \leq 1$, $d_H(a, 001) \leq 1$, or $d_H(a, 110) \leq 1$. Is it possible to cover B_3 with two closed unit disks? Why or why not?

Exercise 5.8 Suppose that we want to cover B_3 with closed disks of radius 2; show that the minimum number of disks needed is 2. In other words, show that the disk

$$D(a_1a_2a_3, 2) = \{a \text{ in } B_3 \text{ such that } d_H(a, a_1a_2a_3) \leq 2\},$$

for any choice of $a_1a_2a_3$ in B_3, will miss at least one point of B_3. Also, you need to show that for any choice of points a, b in B_3, the disks $D(a, 2)$ and $D(b, 2)$ cover B_3.

Exercise 5.9 **What is the minimal number of closed unit disks you need to cover B_4?[2]

Exercise 5.10 ***In general for the set B_n, what is the minimum number of points $a_1, ..., a_k$ such that these points, together with their closed unit disks, cover B_n?[3]

Next, we generalize the Hamming distance to accept sequences of letters, not just 0s and 1s. As above, let's start by defining suitable sets to work with.

The set of English letters will be denoted A, so $A = \{a, b, c, ..., x, y, z\}$. We then define the sets A_n, where n is an integer, via

$$A_n = \{a_1a_2...a_n : a_i \text{ is in } A \text{ for each } i\}.$$

In other words, A_n consists of all sequences of English letters that have length n. For example, $A_1 = A$, the sequence "math" is in the set A_4, and the sequence "asdfghjkl" is in the set A_9. We don't require the elements of A_n to be actual, proper, English words, rather they just need to consist of English letters.

The Hamming distance for sequences of letters is defined analogously as it was for sequences of binary numbers: if a and b are in A_n, then $d_H(a, b)$ is the number of elements in the sequences that disagree. For example, $d_H(\text{cat}, \text{car}) = 1$, since "cat" and "car" only differ in the last letter, but $d_H(\text{cat}, \text{tca}) = 3$, because all of the letters in the sequences differ.

Exercise 5.11 What is the closed unit disk, with respect to d_H, centered at the letter "a"? How many elements are there? What is the open unit disk $D^\circ(a, 1)$?

Exercise 5.12 How many elements are in the closed unit disk centered at "math"?

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[2]One strategy might be to "sketch" the space B_4. One can visualize B_2 as a square, and B_3 as the vertices of a cube, where an edge between elements a and b indicates that a and b differ only in one spot. What would the corresponding sketch be for B_4?

[3]This will be a much more challenging problem.

Word ladders

Here's a fun activity: given two words of the same length, is it possible to transform one into the other by changing individual letters, with the requirement that at each step you still have an English word? For example, to transform a cat into a fish, we might take the sequence

$$\text{cat} \to \text{cad} \to \text{cod}.$$

"Cad" is not an English word[4], so this sequence is not permissible. However, we could use the sequence

$$\text{cat} \to \text{cot} \to \text{cod}.$$

As stated above, we are interested in whether or not two words can be transformed into one another. Moreover, we are interested in the *minimal* number of transformations needed.

Exercise 5.13 Show that "cold" and "warm" can be connected by a word ladder consisting of four steps. How many different word ladders can you come up with?

Exercise 5.14 Show that "ape" and "man" are connected in at most six steps. Can "ape" and "man" be connected in fewer than six steps?

Word ladders are a game invented by Lewis Carroll, and the exercise above comes directly from him; he found that six steps were required to connect "ape" and "man". We can construct a new metric, called the **word ladder metric** and denoted d_{WL}, by the following rule: if w_1 and w_2 are two words of the same length, then

$$d_{WL}(w_1, w_2) = \text{ the length of the word ladder connecting } w_1 \text{ and } w_2.$$

Exercise 5.15 Prove that for any two words w_1 and w_2, we have $d_H(w_1, w_2) \leq d_{WL}(w_1, w_2)$¿

Exercise 5.16 Using the last exercise (or otherwise), show that d_{WL} is in fact a metric on

$$W_n = \{\text{English words of length } n\}.$$

[4]If it is, pretend it isn't.