CHMC: Steiner Symmetrization

6/30/18

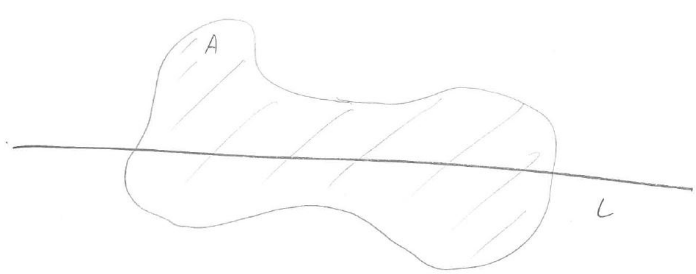
1 Introduction

Steiner symmetrization is a powerful tool in the field of math known as shape optimization. In this worksheet, we'll learn what Steiner symmetrization is, explore some of its basic properties, and see how it connects to shape optimization. In particular, we'll use Steiner symmetrization to give a heuristic proof that among all planar shapes that enclose a given area, a circle has the smallest perimeter.

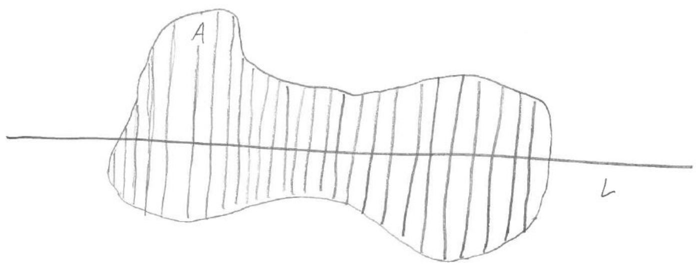
2 Steiner Symmetrization

We'll start first with an example, which we'll use as the effective definition of Steiner symmetrization.

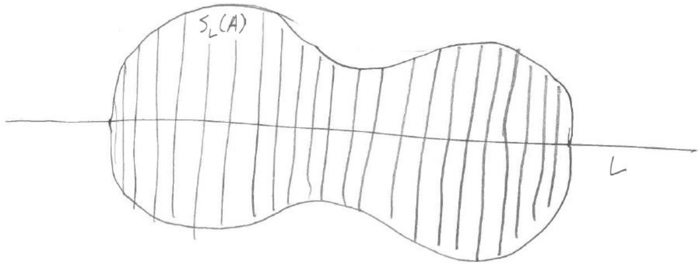
Below, we have a shape A and a line L (part 1). We first slice A into pieces by cutting A with lines perpendicular to L (part 2). Next, we translate all of the pieces so that they are symmetric across L (part 3). This final shape, denoted $S_L(A)$, is called the **Steiner symmetrization** of A. Note that the symmetrization depends on two pieces: the shape A, and the choice of line L.



part 1

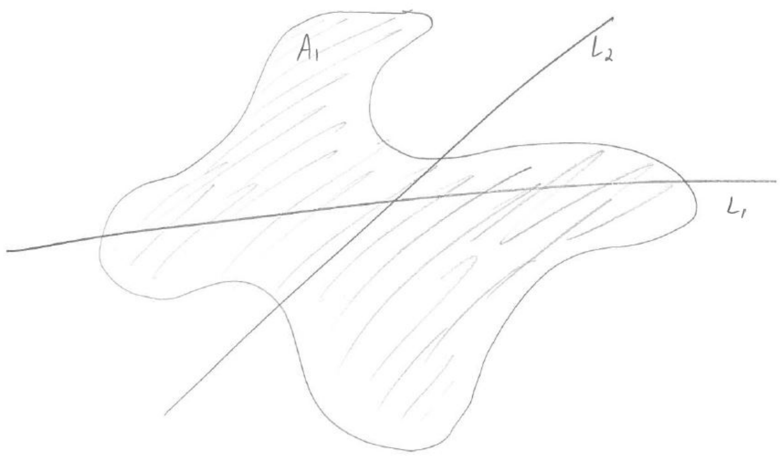


part 2



part 3

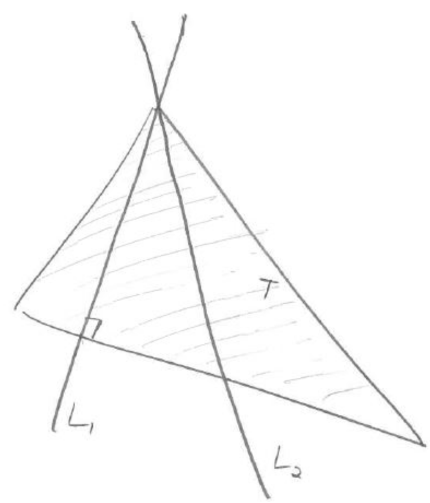
The following figure is for exercises 2.1 and 2.2.



Exercise 2.1 What is the Steiner symmetrization of A_1 with respect to L_1? What about with respect to L_2? Sketches of the shape are perfectly acceptable and don't need to be exact.

Exercise 2.2 What is the Steiner symmetrization of A_1 with respect to *both* L_1 and L_2? In other words, what is $S_{L_1}(S_{L_2}(A))$? Is it the same as $S_{L_2}(S_{L_1}(A))$?

The symmetrizations of general shapes can be hard to draw, so most of this worksheet will focus on polygons. As an easy transition, consider the triangle T and lines L_1, L_2 drawn below.



Exercise 2.3 What is the Steiner symmetrization of the triangle T with respect to L_1? What about with respect to L_2?

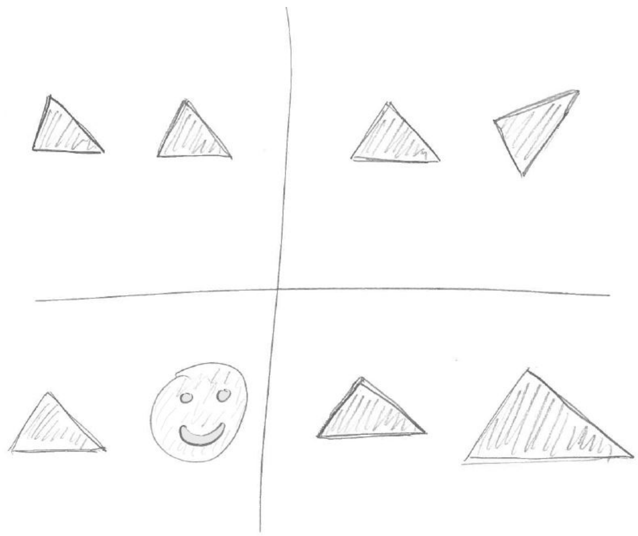
One of the symmetrizations above resulted in another triangle, whereas the other did not. It turns out that any triangle T can be transformed into an equilateral triangle by a sequence of (at most three) Steiner symmetrizations.

Exercise 2.4 Which three lines work to transform T into an equilateral triangle?

Equilateral triangles are usually characterized by having all sides the same length, but now we have another characterization: a triangle is equilateral if it is (Steiner) symmetric with respect to all three of its perpendicular lines. The concept of being Steiner symmetric will be explored in some of the exercises that follow.

One important thing to note is what our notion of "sameness" is. In this case, we call two shapes the same if one is a translate of the other; otherwise they are different shapes. If one shape is a larger (or smaller) version of another, then we do not consider them the same.

Exercise 2.5 Which of the following pairs of shapes are the same, using our notion of sameness? For those that aren't, why not?



With this notion in mind, we can start to study in more detail how Steiner symmetrization behaves with respect to different lines, as well as some other interesting properties of this tool.

Exercise 2.6 Suppose the lines L_1 and L_2 are parallel. If A is any shape in the plane, is it true that $S_{L_1}(A) = S_{L_2}(A)$? Why or why not?

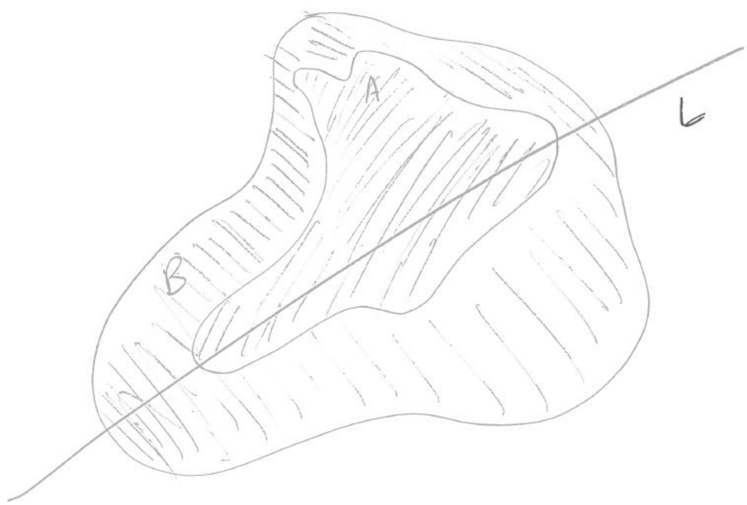
Exercise 2.7 Steiner symmetrization preserves the area of a shape; why? Explicitly, argue that if $S_L(A)$ is the Steiner symmetrization of A with respect to L, then $\text{Area}(S_L(A)) = \text{Area}(A)$.

A related, but **very important**, property that symmetrization has is the fact that the perimeter of $S_L(A)$ will always be less than or equal to the perimeter of A. Symbolically, we express this as $\text{Perim}(S_L(A)) \leq \text{Perim}(A)$. The easiest proof of this fact requires some calculus though, so we'll accept it as a black box for now. [1]

[1] If you know some calculus, feel free to ask about the proof; we'd be more than happy to discuss it.

Another property this symmetrization has is something called **monotonicity**, defined through the next two exercises.

Exercise 2.8 In the figure below are two shapes A and B, and a line L. Sketch $S_L(A)$ and $S_L(B)$; what do you notice?



Exercise 2.9 Suppose A and B are any two shapes, and A is contained completely within the interior of B. If L is any line, then $S_L(A)$ is contained within the interior of $S_L(B)$; why?

A property that shouldn't be too much of a surprise (but is still important) is something called **idempotency**: $S_L(A)$ is the same shape as $S_L(S_L(A))$. To be concise, if A and B are the same shape then we may write $A = B$.

Exercise 2.10 Why is it always true that $S_L(A) = S_L(S_L(A))$?

Exercise 2.11 Now for the general case: if $S_L(A) = A$, what can you conclude about A?

We saw earlier that $S_{L_1}(S_{L_2}(A))$ is not always the same as $S_{L_2}(S_{L_1}(A))$, and we just saw that, say, the shape $S_{L_1}(A)$ is symmetric across the line L_1.

Exercise 2.12 Is it true that $S_{L_1}(S_{L_2}(A))$ is symmetric across the line L_2? Hint: it is not, give an example.

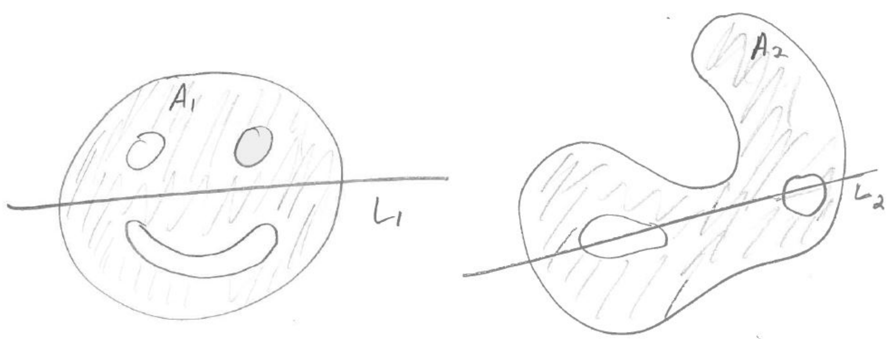
Exercise 2.13 What must be true of the lines L_1 and L_2 to *ensure* the shape $S_{L_1}(S_{L_2}(A))$ is symmetric across L_2?

Exercise 2.14 Give an example of a shape A, and two lines L_1 and L_2, such that $S_{L_1}(S_{L_2}(A))$ is symmetric across both L_1 and L_2, and $S_{L_2}(S_{L_1}(A))$ is symmetric across both L_1 and L_2, but $S_{L_1}(S_{L_2}(A))$ and $S_{L_2}(S_{L_1}(A))$ are different shapes. Hint: let A be a triangle. Can you come up with an example using a quadrilateral?

3 Examples

In this section we'll look at a few more examples of symmetrizing objects, followed by an exploration of how polygons behave with respect to symmetrizations.

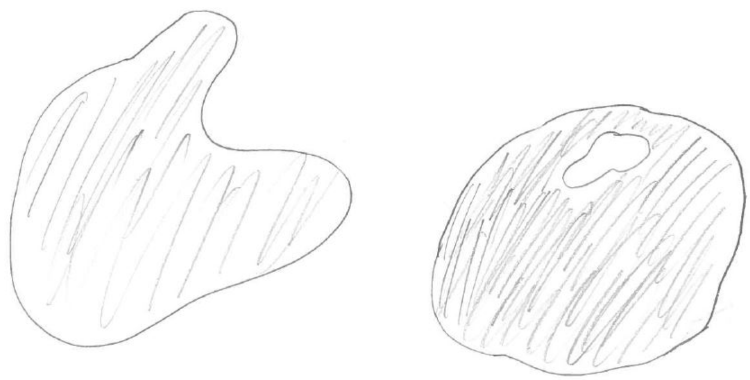
Exercise 3.1 What is the Steiner symmetrization of the shapes A_1 and A_2, with respect to the lines L_1 and L_2?



Exercise 3.2 Is it possible for a shape A to have any "holes" after Steiner symmetrization? Why or why not?

We call a shape A **convex** if any two points in A can be connected by a straight line completely contained within A.

Exercise 3.3 The two shapes below are not convex; why?



While symmetrization will remove any holes in a shape, it may not make a shape convex.

Exercise 3.4 Give an example of a shape A and a line L such that $S_L(A)$ is not convex.

Exercise 3.5 Show that if a shape A is convex, then for any line L we have that $S_L(A)$ is also convex.

The next few exercises should be thought of as more open ended explorations, looking at how Steiner symmetrization affects polygons. You can assume that segments of straight lines will become segments of straight lines after symmetrization.[2]

Exercise 3.6 If T is any triangle, and L is any line, what are the possible shapes that $S_L(T)$ can be? In other words, how many possible edges can $S_L(T)$ have?

Exercise 3.7 Now let Q be any quadrilateral, and L any line. What are the possible shapes that $S_L(Q)$ can be?

Exercise 3.8 Generalize! If P is *any* polygon, what are the possible shapes that $S_L(P)$ can be (depending on your choice of line L)?

Exercise 3.9 How does the pattern change if P is required to be regular? I.e. if P is a regular polygon, and L is any line, what are the possible shapes that $S_L(P)$ can be? Must $S_L(P)$ be regular?

[2] As a challenge, prove this. The trick is coming up with the right framework, but once the framework is there then it's a straightforward computation.

4 The isoperimetric inequality

The full isoperimetric inequality states that, for any bounded (and closed) shape A,

$$4\pi \text{Area}(A) \leq (\text{Perim}(A))^2.$$

Exercise 4.1 For what shape A do we get equality $4\pi\text{Area}(A) = (\text{Perim}(A))^2$?

The theorem we will prove here, due to Steiner (1830), is the following analogue of the isoperimetric inequality:

Theorem 1. *If A is a closed and bounded shape in the plane, and A has as small a perimeter as possible, then A must be a ball.*

Before proving it, though, let's explore what exactly this theorem is telling us.

Exercise 4.2 Suppose you have a rectangle R with area a and width 2. What must the length of R be? What is the perimeter of R?

Exercise 4.3 Generalize the last example: if a rectangle R has area a and width w, what must the length of R be? What is the perimeter of R? How would you chose the width to make the perimeter as large as possible?

Exercise 4.4 If a rectangle R has area a, what do the side lengths of R need to be to minimize perimeter? Can you prove it?

The last few exercises show that, for a given area, there is no upper bound on the perimeter of a shape. Therefore trying to maximize perimeter is not so interesting, but minimizing perimeter is.

The idea of Steiner's proof is the following: if A is any shape in the plane, and L is any line, then $\text{Area}(A) = \text{Area}(S_L(A))$ but $\text{Perim}(A) \geq \text{Perim}(S_L(A))$. Thus, a shape that has the smallest perimeter for a fixed area should be (Steiner) symmetric with respect to any line. The only such shape is a circle, so amongst all shapes that enclose a fixed area the circle has the smallest perimeter.

There's a slight issue with this proof. What we have is a method of constructing a new shape with a smaller perimeter from an old one, but there's no guarantee that this process will ever stop. An analogy is given by the following argument that 1 is the smallest positive real number.

Let $x \neq 1$ be a positive real number, so that $x^2 - x = x(x-1) \neq 0$ (why is this true?). If $x^2 > x$, then $x(x-1) > 0$, so $x > 1$. Thus, 1 is smaller than our arbitrarily chosen positive real number x (i.e. 1 is already proving to be the smallest positive number).

If instead of $x^2 > x$ we have $x^2 < x$, then we have a method of producing a smaller positive real number than x, in this case x^2.

Thus, 1 is the smallest positive real number.

Exercise 4.5 Why is this not a proof that 1 is the smallest positive real number?

The connection to Steiner's argument is made by thinking of the smallest positive real number 1 as the smallest perimeter shape a circle, and thinking of $x^2 < x$ as the inequality $\text{Perim}(S_L(A)) \leq \text{Perim}(A)$. Thus, for Steiner's proof to be complete we need assurance that a limit shape does actually exist.

If you take any shape A, and any sequence of lines $L_1, L_2, ...$, then Steiner's argument is equivalent to constructing the shape $S_{L_n}(...S_{L_2}(S_{L_1}(A))...)$ as n tends to infinity. Intuitively we would like to say the limiting object is always a circle, but a priori that doesn't need to be true. Let's look at a special case of this.

Let L be any line, A any shape, and R a rotation by θ degrees (you can assume L passes through the origin and A is centered at the origin). Define $L_1 = L, L_2 = RL_1, L_3 = RL_2$, etc., so that L_n is the line L rotated by $(n-1)\theta$ degrees. Must the shape $S_{L_n}(...S_{L_1}(A)...)$ converge to a circle? Not necessarily!

Exercise 4.6 Give an example of a shape A, a line L, and a rotation R, such that the limit of the shapes $S_{L_n}(...S_{L_1}(A)...)$ is not a circle.

It is true, though, that if your rotation R is by an irrational angle θ, then the limiting object *will* be a circle. The proof of this fact goes far beyond the tools we have at hand.