CHMC Advanced Group:
Taxicab Geometry

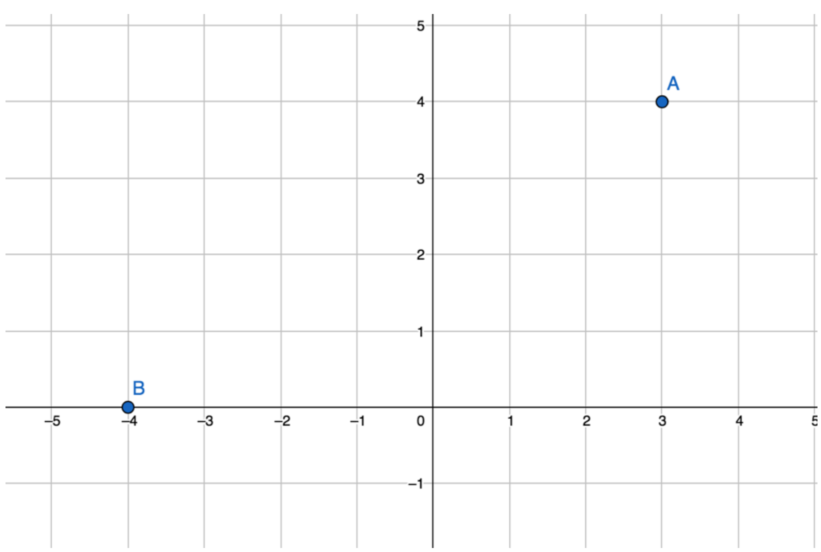
06/16/2018

1 Introduction

In school we are taught "geometry," which looks at objects (such as points, lines, shapes) in a plane and determine relationships between them. It turns out that this is only one of many types of geometry. We call this "Euclidean geometry," named after the mathematician Euclid, who came up with it. In fact, Euclid wrote several books, including "Elements," in which he wrote many of the theorems that we know today, such as: Every triangle has an angle sum of exactly 180°.

Today we will explore one of the other types of geometry, called "taxicab geometry. Much like all of Euclidean geometry happened in the plane, taxicab geometry also happens in the plane. The difference is in how we calculate distance. In taxicab geometry, we measure distance as if we are in a taxi and must travel along roads, which go only left/right or up/down, in order to get to our destination. In Euclidean geometry, distance is measured as though we are a bird that can fly directly between any two points.

Exercise 1.1 What is the taxicab distance between points (3,4) and (-4,0)? What about the Euclidean distance?



2 Taxicab Metric

For two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane, the **Euclidean** distance is defined as

$$D(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

In the following exercise, you will write down an explicit formula for the taxicab metric in the plane.

Exercise 2.1 Write a formula for the taxicab distance between any two points (a, b) and (c, d). Think of how you would walk from (a, b) to (c, d) where the only valid ways to move are left\right and up\down.

Exercise 2.2 Calculate the Euclidean distance and the taxicab distance for the following pairs of points:

i) $(0, 0)$ and $(5, 0)$

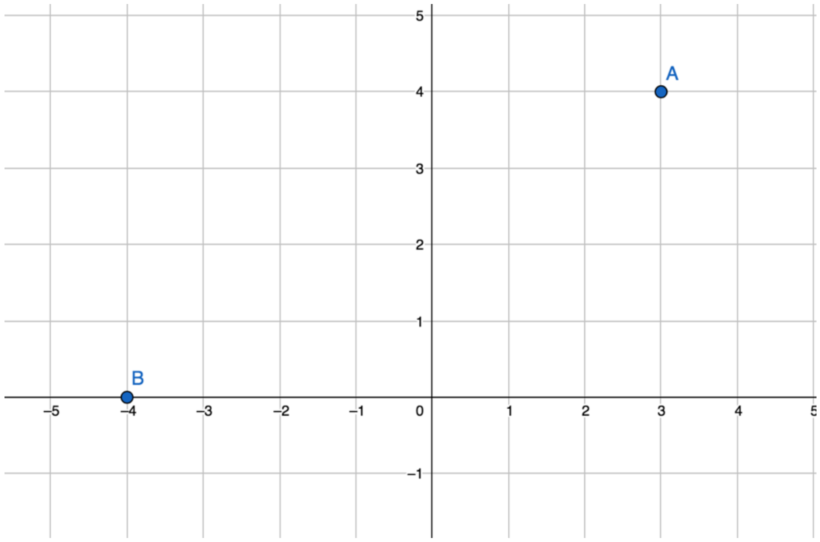
ii) $(0, 0)$ and $(3, 4)$

iii) $(1, 4)$ and $(13, 9)$

Exercise 2.3 In general, how will the Euclidean distance between points compare to the taxicab distance? When do the two metrics coincide, that is, when do the Euclidean and Taxicab metrics agree for a pair of points (a, b) and (c, d)?

A **geodesic** is a line segment that goes along the shortest path between any two points. In Euclidean geometry, geodesics are unique, meaning that there is only one such segment between any two points (this is the line going between them).

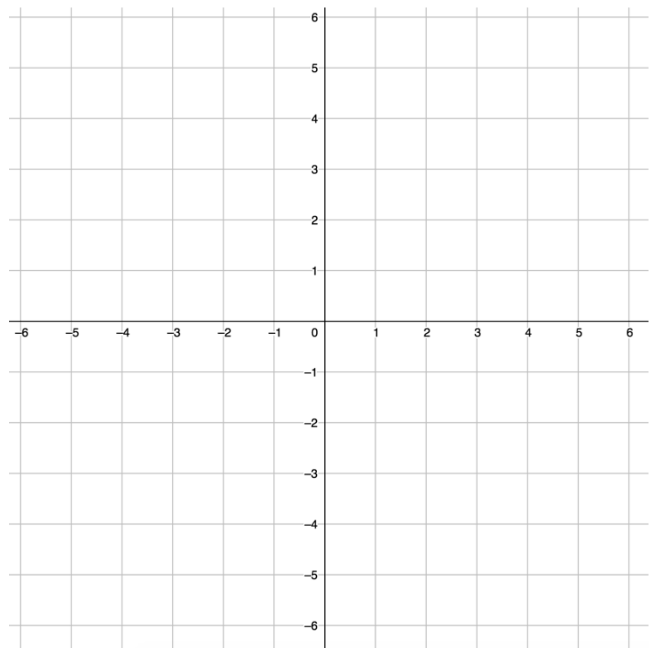
Exercise 2.4 Are geodesics unique in taxicab geometry? (Draw some examples between A and B)


A
B
5
4
3
2
1
0
−1
−5
−4
−3
−2
−1
1
2
3
4
5

3 Geometry with the taxicab metric

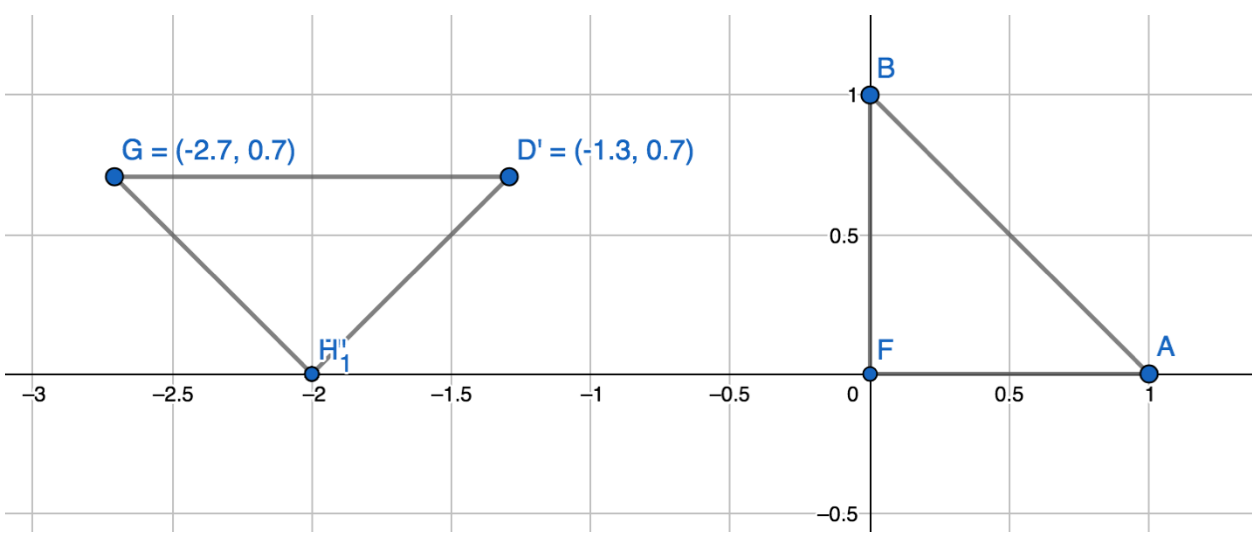
Recall that a circle is defined by the set of all points that are r units away from the center point, i.e., r is our radius where $r > 0$.

Exercise 3.1 Draw a taxicab circle with a radius and center point of your choosing, i.e., draw the set of points that will be r taxicab units away from your center point. What is the shape of the circle?



Below are two congruent triangles in Euclidean geometry.

Exercise 3.2 Find the side lengths of these triangles in Taxicab geometry. Are the triangles congruent in taxicab geometry?

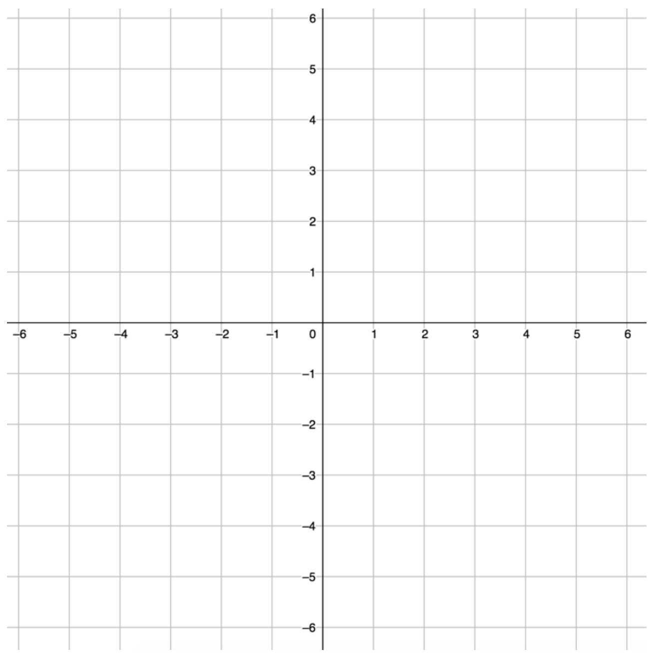


Exercise 3.3 Describe how to get from triangle ABF to triangle DGH using only rotations and translations.

Exercise 3.4 Does translating a triangle impact its taxicab side lengths? What about rotation?

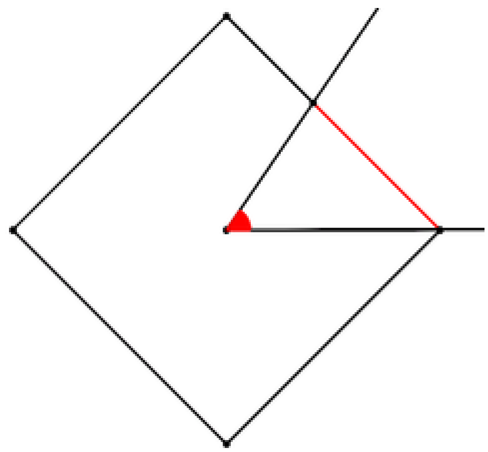
Exercise 3.5 How does reflecting a taxicab triangle over a horizontal or vertical line change the side lengths? What about other types of lines?

Exercise 3.6 In Euclidean geometry, the sum of any two side lengths must be greater than the third side length. Does this rule still hold in taxicab geometry? Try to find a counter-example.



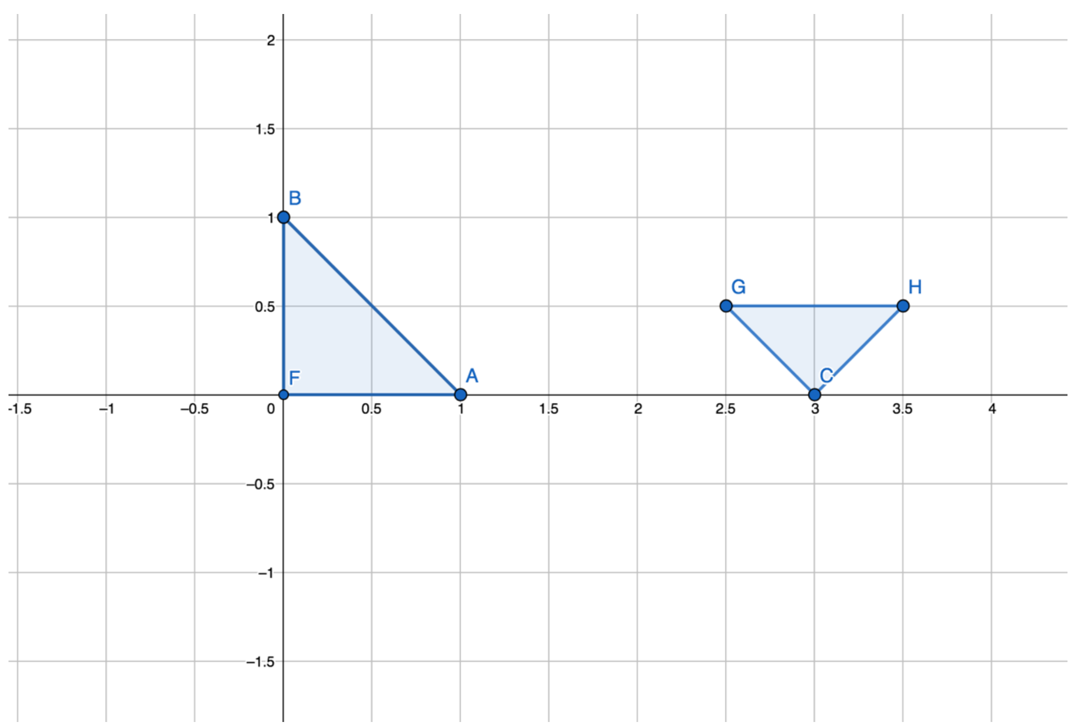
4 Angles in taxicab metric

Recall that, in Euclidean geometry, triangles have an angle sum of 180°. We want to find the analogous statement for triangles in taxicab geometry. In taxicab geometry, angles are measured in "taxicab radians," or "t-radians." One **t-radian** is an angle whose vertex is at the center of a unit taxicab circle and intersects an arc of length 1, as shown below:



Exercise 4.1 In the above picture, we see that the red angle has measurements of 45° in Euclidean geometry and 1 t-radian in taxicab geometry. Will any 45° have taxicab measurement of 1 t-radian?

Exercise 4.2 Calculate the angle sum of triangles below:



We can also explore using taxicab distance in three dimensions. Now, our taxi is allowed to move East\West, North\South, and up\down (into the air or into the ground!).

Exercise 4.3 Write a formula for the taxicab distance between any two points (a, b, c) and (d, e, f).

Exercise 4.4 A sphere of radius r is defined to be the set of all points that are r units away from the center point. What shape is the taxicab sphere that is centered at the origin and has radius 1?

Exercise 4.5 Consider a cube whose side lengths are each 1 (in Euclidean measure). Place the cube in three dimensional space so that 3 of its sides go along the axes. More precisely, place one of the vertices of the cube at the origin and the three edges from that vertex along the positive $x-$axis, $y-$axis and $z-$axis. What is the taxicab volume of this cube? Now rotate the cube $45°$ in the xy-plane, so that one face of the cube is in the $xy-$plane and with a vertex on the positive $x-$axis and another vertex on the positive $y-$axis. What is the taxicab volume of the rotated cube?

Exercise 4.6 How should one rotate the cube in order to maximize its taxicab volume?

Exercise 4.7 In $n-$dimensional space, where points are denoted by $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ how do you think the taxicab metric is defined?