CHMC Advanced Group: Axiomatic Systems

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1 A Crisis of Foundations

Modern mathematics is based on the concept of axiomatic systems. During the 19th and 20th centuries, mathematicians began to notice problems and paradoxes that resulted from shaky foundations. As a result, it became a high priority to make concrete and definite sense of the work that they were doing. The solution they came up with was to base mathematics on axiomatic systems. Today, we will look at what an axiomatic system is and will study several thought-provoking examples. Studying axiomatic systems helps us to better understand and appreciate the math that we do. It also trains us to better understand the logical reasoning that is used in conversations and arguments in our non-mathematical lives, enabling us to more wisely make decisions and evaluate the information we receive. Before we get in to exactly what axiomatic systems are, let's spend a little bit of time understanding the need for them.

2 Why do we need axioms?

As humans, we operate in and interact with the world primarily by taking in information and filtering it through a framework built from our experiences and assumptions. This is the way that we make decisions, all the way from what college attend to what kind of clothes to wear. The decision making process illustrates the use of axioms in our everyday lives. An axiom is a statement that is taken to be true, and it is used as a starting point for arguments. Our experience in the world, for example, has taught us that if we have two apples and then we obtain two more, we will have a total of four apples. So when we see the equation $2 + 2 = ?$ on a quiz, we happily and confidently write down 4.

However, something of crucial importance has happened between the act of counting apples and the act of figuring out this sum. We have assumed that the principle we observed when we had some apples and obtained some more would also hold if, say, we had pears instead. Actually, we have assumed more than this. We have assumed the principle also would hold if we were counting books, shoes, pencils, or any other object. We have made the assumption that the "apple counting principle" holds in an absolute sense, entirely independent of what objects I am counting.

Actually, we have done even more. We have assumed that there is something definite and well-defined that we are referring to when we write 2. And when we add it to 2, we have assumed that there is something definite and well-defined called 4. We are assuming that there are objects known as non-negative whole numbers, and that we can add two of them together to obtain another non-negative whole number in a meaningful manner that is consistent with our intuitive ideas about "two-ness", "three-ness", etc.

This may seem technical, but it is extremely important for the way that we do mathematics. We commonly think of mathematical statements as being either "right" or "wrong", "true" or "false", but in order for us to do so, we must actually speak about concrete mathematical realities in a meaningful way. This is where axiomatic systems come in.

To summarize, here's what we have so far:

1. Our assumptions shape how we think about and interact with the world.

2. Our experiences motivate how we do mathematics.

3. Our assumptions are indispensable for assigning meaning to mathematical statements.

4. $2 + 2 = 4$ is actually a very profound statement.

3 What is an axiom?

Now that we've seen how even the simplest mathematical statements depend on our assumptions, it is time for us to define axioms and to look at what it looks like to do mathematics in an axiomatic system. An **axiom** is precisely an assumption: it is a statement that we take to be true without prior justification. Axioms are our starting points when doing mathematics. We simply declare them to be true, and the consequences of them, or the results that we can deduce from them, are called **theorems**. An **axiomatic system** is a collection of axioms that can be used along with the rules of logic to deduce theorems.

For example, I may take the statement *"the earth is round"* to be an axiom. As a consequence of this axiom, I could deduce that the earth is not flat. So in this axiomatic system, the statement *"the earth is not flat"* is a theorem; that is, it is a consequence of our axiom.

In some other axiomatic system, I might take the statement "*the earth is flat*" to be an axiom. In this system, the statement "*the earth is not round*" could be deduced as a theorem. This is a perfectly acceptable axiomatic system, though we might as first be opposed to it, since we all know today that the earth is not flat, but is indeed round.

Or is it? When viewed on a large scale, the earth is indeed round. However, when we zoom in, we see that the earth's surface is full of oceans, valleys, mountains, trees, rocks, buildings, and people; all of which make the earth's surface very far from being round. If we zoom in even further and restrict our attention, say, to a basketball court, then we would observe that the earth is quite flat indeed. If I am playing basketball, then the assumption that the earth is flat is a far more relevant and true statement for the purposes of dribbling and moving on the court.

This reveals a very important aspect of choosing axioms. In an axiomatic system, we are much more concerned with things being *consistent* than we are with them being *true* in an absolute sense. Consistency means that no statement in the system can be proven to be both true and false. If we have statements that are both true and false, then we have ceased being able to speak meaningfully at all.

Furthermore, the example of the earth reveals the importance of definitions in an axiomatic system. The only reason why the statements "*the earth is flat*" and "*the earth is round*" are opposed to each other is because we have a notion of what flatness and roundness is. You can try to give these statements precise mathematical definitions, but you will likely find that in doing so, you have introduced more words that you have not yet defined. In an axiomatic system, we sometimes choose to leave some terms undefined, and we understand their meaning primarily in terms of the axioms given, even though they might be intended to refer to something intuitive. We will see examples of this below.

4 Euclid's Axiomatic System [2]

Although axiomatic systems were used in the 19th and 20th centuries to put mathematics on a firm foundation, their use actually goes back several thousand years with what is probably the most familiar axiomatic system: Euclidean Geometry. Sometime around 300 B.C. Euclid began his geometric explorations with 23 definitions, five common notions, and five axioms (sometimes called postulates). Euclid's five axioms are

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. [1]

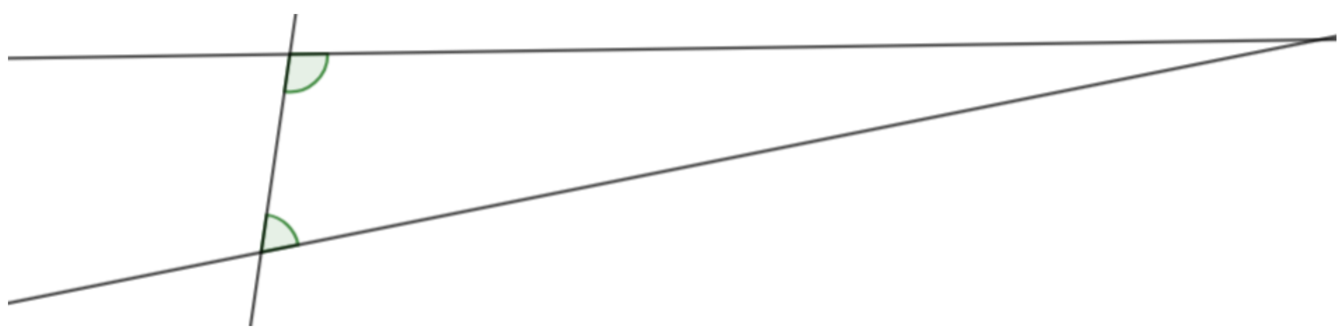

Figure 1: Since the sum of the two angles shown is less than two right angles, the 5th axiom tells us that the two lines intersect on the right side of the vertical line.

Exercise 4.1 What terms in the axioms above need definitions? How would you define them? Are there any terms you would choose to leave undefined? Compare your choices with those of someone around you.

As we discussed above, axioms are statements that we accept as true and use as starting points for our further investigation. However, generally we do not just choose our axioms arbitrarily. One of the reasons why Euclidean Geometry is still taught in schools today is because of the intuitive nature of the axioms. They seem reasonable, and they also fit well with our daily experience. Furthermore, with the exception of possibly the 5th axiom, they are simple statements and feel fundamental and basic to us. While mathematicians do not have rules for how we choose axioms, it is generally accepted that we ought to choose them to be intuitive, basic, and reasonable if we want the theorems that we prove to be meaningful.

Let's explore some Euclidean Geometry. In the exercises below, it is helpful to draw pictures of the situations, but make sure to justify each claim that you make using the axioms above. Unfortunately, we don't have time here to defined precisely all of the terms that appear below, but just interpret them in the usual sense.

Exercise 4.2 Prove that it is possible to construct an equilateral triangle, one of whose edges is a given line segment. Each step in your construction could be done using a compass and straightedge, but make sure to use the axioms to justify why each step of the construction is possible.

Euclid's second theorem states that given a line segment L and a point A not on L, it is possible to construct a line segment through A that has the same length as L. Use this theorem for the next exercise.

Exercise 4.3 Given two straight line segments L_1 and L_2, with L_1 of greater length than L_2, prove that it is possible to cut off from L_1 a line segment with length equal to L_2. This means that we can divide L_1 into two line segments, one of which has the same length as L_2.

Exercise 4.4 Euclid's 32nd theorem states that the sum of the angles in a triangle is exactly two right angles. Use this to prove the following result: "If a line transversing two other lines makes the alternate interior angles equal to each other, then the two lines are parallel."

Exercise 4.5 Euclid's 12th theorem states that given a line L and a point P not on L, it is possibly to draw a line through P perpendicular to L.

Use this theorem, along with Euclid's axioms and the previous exercises to prove that given a line L and a point P not on L, there is exactly one line through P parallel to L. This result is known as Playfair's axiom (though in this case it is a theorem, and not an axiom), and it is actually equivalent to Euclid's 5th axiom.

5 A Simpler Universe

We observed above that Euclid's axioms are intuitive and fit well with our everyday experience. But since we recognize that we are simply assuming these axioms to be true, it is natural to wonder what would happen if we replaced them with other plausible or interesting assumptions.

A **projective plane** is a set of points and lines, along with a notion of "incidence" where we are assuming the following as axioms

1. Given two distinct points, there is exactly one line incident with both of them.

2. Given two distinct lines, there is at least one point incident with both lines.

3. Every line has at least three points incident with it.

4. There exist four distinct points so that no line is incident with more than two of them.

In this axiomatic system, points, lines, and the notion of incidence are left as undefined terms. However, it is helpful to think of them as geometric objects.

In Euclidean Geometry, there are an infinite number of points, and lines can be extended indefinitely. In a projective plane, however, we don't know from the outset how many points and lines there actually are. In fact, the best we can say at this point is that the 4th axiom guarantees the existence of four points. There could actually be various different collections of points and lines that satisfy the axioms, which is why we say *a* projective plane rather than *the* projective plane. We will explore this below.

Exercise 5.1 Axiom 2 tell us that there is at least one point incident with any two distinct lines. Use the axioms above to prove that there is exactly one. Hint: if you assume that there are two points incident with two distinct lines, can you obtain a contradiction to the axioms?

Exercise 5.2 Replace axiom 1 by the statement "Given two distinct points, there is at least one line incident with both of them." Replace axiom 2 by the statement "Given two distinct lines, there is exactly one point incident with both of them." Now prove the statement "Given two distinct points there is exactly one line incident with both of them."

The previous two exercises express a property known as the "duality of the projective plane." If we interchange the words "point" and "line" in our axioms, we actually have the exact same axiomatic system! So even though the axioms for a projective plane might not be as intuitive as Euclid's axioms, they lead to quite a simple universe, one in which there is a precise symmetry between the properties of points and lines.

Exercise 5.3 In Euclidean Geometry, there are an infinite number of points and lines. Can you find an example of a projective plane that has a finite number of points? Your answer should be a list of points and lines, along with the data of what points are incident with each line, and vice versa. It might be helpful to draw a picture. What is the fewest number of points that a projective plane must have? Prove your findings using the axioms and the previous exercise.

Exercise 5.4 The previous exercise studied a finite collection of points and lines that satisfy the axioms of a projective plane. Can you think of an infinite collection of points and lines that satisfy these axioms? Use your imagination, and carefully verify that the axioms hold.

Different kinds of geometry like the one explored here challenge us to use our creativity and imagination to get a handle on what worlds with those different kinds of geometry could be like. In fact, understanding that Euclidean Geometry is based axioms begs the question: how do we even know that our own world obeys the theorems and results proven there? Our experience tells us that lines, distances, and angles in our world behave as they do in Euclidean Geometry; however, we humans experience the universe on a relatively small scale. Perhaps our world only appears to be Euclidean on small scales, but if we could, say, measure a triangle with endpoints at the center of three different galaxies, we might find that the angles do not add up to 180 degrees. In fact, Einstein's General Theory of Relativity has obtained very accurate predictions of physical phenomena under the assumption that our world is not Euclidean. But alas, this is a topic for another day.

6 How do you choose axioms?

We have seen a few examples of axiomatic systems so far. The axioms of Euclid are intuitive, and are based on what we observe in our world. The axioms of a projective plane, on the other hand, are somewhat less intuitive. Why should we choose one set of axioms over another? Does this mean that mathematics is all just the consequences of arbitrarily chosen assumptions? In one sense, yes. The way that we understand mathematics today is as a system based on some axioms that we have chosen. On the other hand, our choice of axioms is far from arbitrary, and we generally desire them to express the *fundamental aspects* of what our intuition and experience leads us to desire to study . And we want them to do so in *short* and *simple* statements. We certainly could study arbitrary axioms and call it mathematics, but this is rarely done in practice because we want the mathematics that we study to have meaning and to fit consistently into the framework of all the mathematics that has been studied up to today.

Beyond this, there are three "rules" that we would like to abide by when choosing axioms.

1. **Consistency**. We do not want our axioms to contradict each other. It is useless to study mathematics in which every pair of points can be connected by a line, and no pair of points can be connected by a line. In fact, mathematician Kurt Gödel proved in the mid 1900s that in an axiomatic system with inconsistent axioms, every statement can be proven to be both true and false. We definitely want to avoid this.

2. **Independence**. We do not want our axioms to depend on each other. We would like for each axiom to express a different fundamental property of whatever it is we are trying to study. If we took a theorem in Euclidean Geometry and added it to our collection of axioms, then we would have not have an independent collection of axioms. This is not quite as important as consistency, but it is reasonable to try not to be redundant. It is, however, quite hard to prove that an axiomatic system is independent. For centuries, people believed that Euclid's 5th axiom could be proven as a consequence of the other four. In the early 1800s, mathematicians Nikolai Lobachevsky and János Bolyai gave examples of collections of points of lines that satisfied Euclid's first four axioms, but not the 5th, thus proving that the 5th was an independent axiom.

3. **Completeness**. Given a true statement, we would like to be able to conclusively prove that it is true. This property is known as the completeness of an axiomatic system. It is not as important as consistency, but mathematics is about proving statements and answering questions, so it would be nice if our axioms enabled us to do this all of the time.

7 Back to Arithmetic

At the beginning of this lesson, we discussed the need for axioms in arithmetic. We saw how the equation $2 + 2 = 4$ expresses a principle that we know to be true in the world of our physical experience. In order to make it a statement that has actual mathematical meaning, we must have some axioms here that dictate the rules for arithmetic.

As a starting point, we could simply assume that the objects we know as non-negative whole numbers exist, and we could write down all the rules telling us how to add them (e.g. $(a + b) + c = a + (b + c)$, $a + b = b + a$, $1 + 1 = 2$, $1 + 2 = 3$, $1 + 1 + 1 = 3$, $2 + 1 = 3$, $1 + 1 + 1 + 1 = 4$, $2 + 2 = 4$, $1 + 3 = 4$, $2 + 3 = 5$, and so on). However, this would give us an infinite collection of axioms, and it would be incredibly difficult to check whether or not they are consistent.

Exercise 7.1 Prove that the infinite collection of axioms for arithmetic given above do not form an independent collection of axioms. Can you find a way to systematically delete some of these axioms to obtain an independent infinite collection?

The above axioms are not satisfactory. Axioms should be fairly simple and intuitive. Furthermore, they should express fundamental properties of what we are dealing with. The above example is similar to declaring every single theorem in Euclidean Geometry to be an axiom. It seems to be unnecessary and is definitely not helpful.

In 1889, mathematician Giuseppe Peano beautifully boiled down the properties of non-negative whole numbers to a total of 9 axioms.

Exercise 7.2 *Before you read on*, spend some time thinking about the non-negative whole numbers. What do you think are some fundamental properties of them? Can you think of ways to express them as axioms?

8 Peano's Axioms

What did you come up with as being fundamental properties of non-negative whole numbers in the previous exercise? Is it the ability to add two of them together to obtain a third? Is the fact that are infinitely many of them? Perhaps it is that we can always compare two numbers and determine which one is bigger? Or maybe you came up with the trichotomy law, that given positive whole numbers n an m, either $n > m$, $n < m$, or $n = m$? Or perhaps you thought a property involving multiplication was more fundamental? Or something involving prime numbers?

Here is what Peano came up with. We will refer to non-negative whole numbers as the "natural numbers" and will denote them by $\mathbb{N}$. The axioms also dictate the properties of equality, written $=$.

1. 0 is a natural number

2. If x is a natural number, then $x = x$.

3. If x and y are natural numbers and $x = y$, then $y = x$.

4. If x, y, and z are natural numbers and $x = y$ and $y = z$, then $x = z$.

5. For any x (x does not have to be a natural number, it can be anything), if y is a natural number and $x = y$, then x is also a natural number.

We also assume that there is a function S, known as the successor function, that takes any natural number n as its input that spits out an object $S(n)$ that satisfies the following three properties:

6. $S(n)$ is a natural number.

7. For any two natural numbers n and m, $n = m$ if and only if $S(n) = S(m)$.

8. For any natural number n, $S(n) = 0$ is always a false statement.

9. If K is a collection of objects so that

   (a) 0 is in K, and

   (b) if n is in K, then $S(n)$ is in K then K contains every natural number.

Exercise 8.1 That was a lot! Take a deep breath.

Exercise 8.2 Read axioms 1-4 again slowly. Try to understand each one before going on to the next.

Axiom 1 declares the existence of something called 0. This is quite simpler than what we proposed in the previous section, where we considered taking it as an axiom that every natural number exists. It turns out we only need to assume that one number exists. Axioms 2-4 discuss the notion of equality. The first property is very basic: of course every number is equal to itself. Axioms 3 and 4 are similarly basic, so much so, that it is likely that you and I would simply have overlooked them without even realizing it. But basic and fundamental is precisely what axioms are supposed to be.

Exercise 8.3 Read axioms 5-8 again slowly. Try to understand each one before going on to the next.

Axiom 5 says that anything that is equal to a natural number must itself be a natural number. So if we write $5 =!!\&-J$, then $!!\&-J$ must be a natural number. Axiom 6 declares that given any natural number, there is a notion of its successor, or the "next" natural number after it. Given this, we understand that 1 is the successor of 0, or $S(0) = 1$. Also, $S(1) = S(S(0)) = 2$, and so on. We can now define any number. For example, we define 5 to be $S(S(S(S(S(0)))))$. To summarize, axiom 1 asserts that 0 exists, and we can use this along with the S function that is assumed to exists to *define* every other natural number.

Axiom 7 says that if two natural numbers are equal, then the next natural numbers after each of them are also equal. It also says that if two numbers have the same successor, then they are in fact the same numbers. Axiom 8 declares that 0 is the first natural number. There is no number that comes before it.

Exercise 8.4 Read axiom 9 again slowly.

Axiom 9 is known as the principle of induction. It says that if K is a set of objects containing 0 so that if any number is in K, then the successor of the number is also in K, then K must contain every natural number. This is the most complicated of the axioms to understand.

Exercise 8.5 Use axiom 9 to prove that every nonzero natural number is some successor of 0.

Exercise 8.6 What kind of number system do you get if you remove axiom 8? Can you come up with a collection of objects that satisfies all of the Peano axioms except for axiom 8, and is different from the natural numbers? Hint: try thinking about the arithmetic of a clock.

These are the ideas that Peano found to be most fundamental about the natural numbers. As we can see, it is quite difficult to systematically come up with axioms that sufficiently describe what it is we would like to study. It is worth noting that the Peano axioms do not ever mention something known as addition. Actually, using the notion of equality along with the S function, we construct addition as a consequence of the 9 Peano axioms. From this, we can define multiplication, as well as the other familiar operations on natural numbers.

These axioms are of crucial importance to mathematics. In fact, from them we could construct the notion of the integers, the positive and negative whole numbers, as well as the rational numbers, and the real numbers, and finally, the complex or imaginary numbers.

Mathematicians have proven the consistency, as well as the independence of the Peano axioms. The third desired attribute of axioms, completeness, was actually proven by Kurt Gödel to be false for the Peano axioms. The importance of this in mathematics cannot be overstated. Essentially, Gödel proved that if we want to be able to do mathematics where we can talk meaningfully about the notion of a number (and this certainly seems like something we would want mathematics to be able to do), then we must accept the reality that some statements that are true cannot be proven to be true, and some statements that are false cannot be proven to be false. This is known as Gödel's Incompleteness Theorem.

9 What should you take away from this

Here are a few things that you should take away from today's lesson.

1. You cannot have modern mathematics without axioms. They are our starting points, and they enable us to give precise mathematical meaning to terms and statements. When you learn new ideas in your math courses, try to think about what assumptions you must make in order to talk about these ideas. This allows you to better understand the math that you do, and it also helps to develop the habit of thinking critically about the things that you are told.

2. In order to speak meaningfully about things, we must have some assumptions. This is a truth that extends beyond mathematics and into the realm of all meaningful human discussion.

3. While mathematics can sometimes speak more definitively than other subjects about what it true, the truth of a mathematical statement is a relative truth; that is, it depends on the axioms that you choose to take as your starting point.

4. Mathematics does not perfectly describe the universe that we live in. Although it gives very accurate predictions for a large number of physical phenomena, it is based on assumptions that are not so much *true* or *false* as they are *useful*. So while mathematics is a type of human knowledge, it is not the end of all human knowledge. Other forms of learning, like science, philosophy, religion, and the arts all have their place in the world, and as people interested in mathematics, we would do well to respect them.

5. Uncertainty is inevitable in mathematics. In order to do the most basic type of mathematics (counting and basic arithmetic), we must make assumptions that introduce uncertainty to mathematics. There are mathematical statements that are true that cannot be proven to be true. Mathematics cannot answer every question that we might ask, and we actually have no way of knowing what questions it can and cannot answer.

References

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