The Number 2 and Graphs

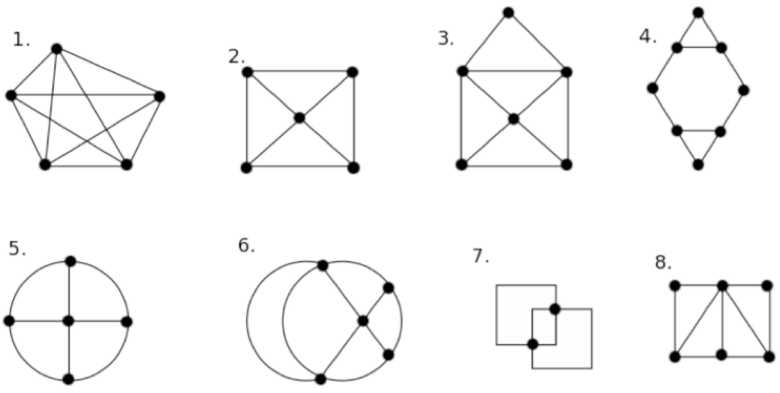
1 Evil Wizard

1. An evil wizard has imprisoned 64 math circle participants. The wizard announces, Tomorrow I will have you stand in a line, and I will put a hat on each of your heads. The hat will be colored either white or black. You will be able to see the hats of everyone in front of you, but you will not be able to see your hat or the hats of the people behind you. I will begin by asking the person at the back of the line to guess his or her hat color. If the guess is correct, that person will get a cookie. If the guess is wrong, that person will be killed in a painful way. Then I will ask the next person in line, and so on. You are only allowed to say the single word black or white when it is your turn to speak, and otherwise you are not allowed to communicate with each other while you are standing in line. Although you will not be able to see the people behind you, you will know (by hearing) if they have answered correctly or not.

   The prisoners are allowed to chat for a few minutes before their ordeal begins. What is the largest number of prisoners that can be guaranteed to survive?

2 Graphs and Paths

2. Try to trace each figure without lifting your pencil from your paper and without going over the same line twice.

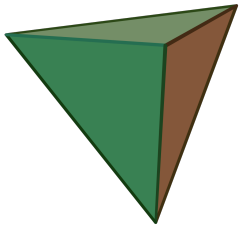


   How can you predict whether it will be possible?

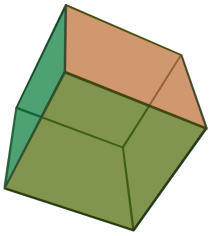
   Note: figures like these made of dots and (possibly curved) line segments is called a *graph*. The dots are called the *vertices* and the line segments are called *edges*.

3. For each of these polyhedra, try to make the edges of the polyhedron out of the smallest number of pipe cleaners possible. You are not allowed to cut the pipe cleaners and you are not allowed to double edges (that is, each edge just just be one pipe-cleaner thick).

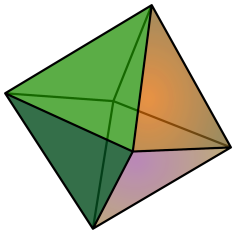
   (a) The edges of a tetrahedron

   (b) The edges of a cube

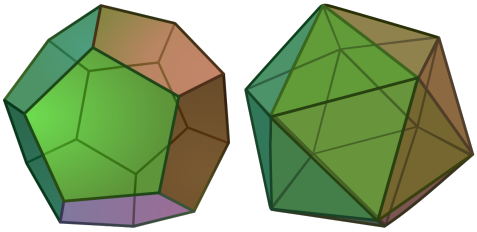
   

   (c) The edges of an octahedron

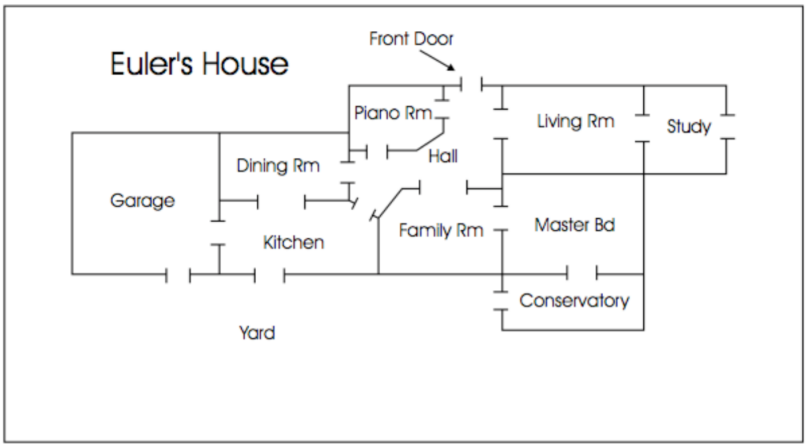
   

4. If a graph has 6 vertices that each meet an odd number of ends of edges, what can you say about the number of pipe cleaners required to make such a graph?

5. How many pipe cleaners would you need to build a dodecahedron? An icosohedron?



6. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in. Will baby Euler succeed? What if the front door is closed?



7. Can you trace the figure without picking up your pencil or drawing over any line twice?

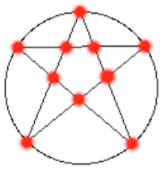


figure 1

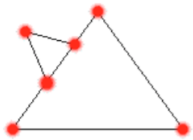


figure 2

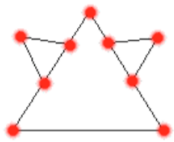


figure 3

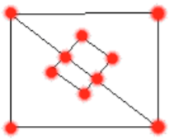


figure 4

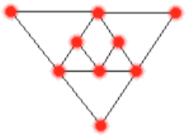
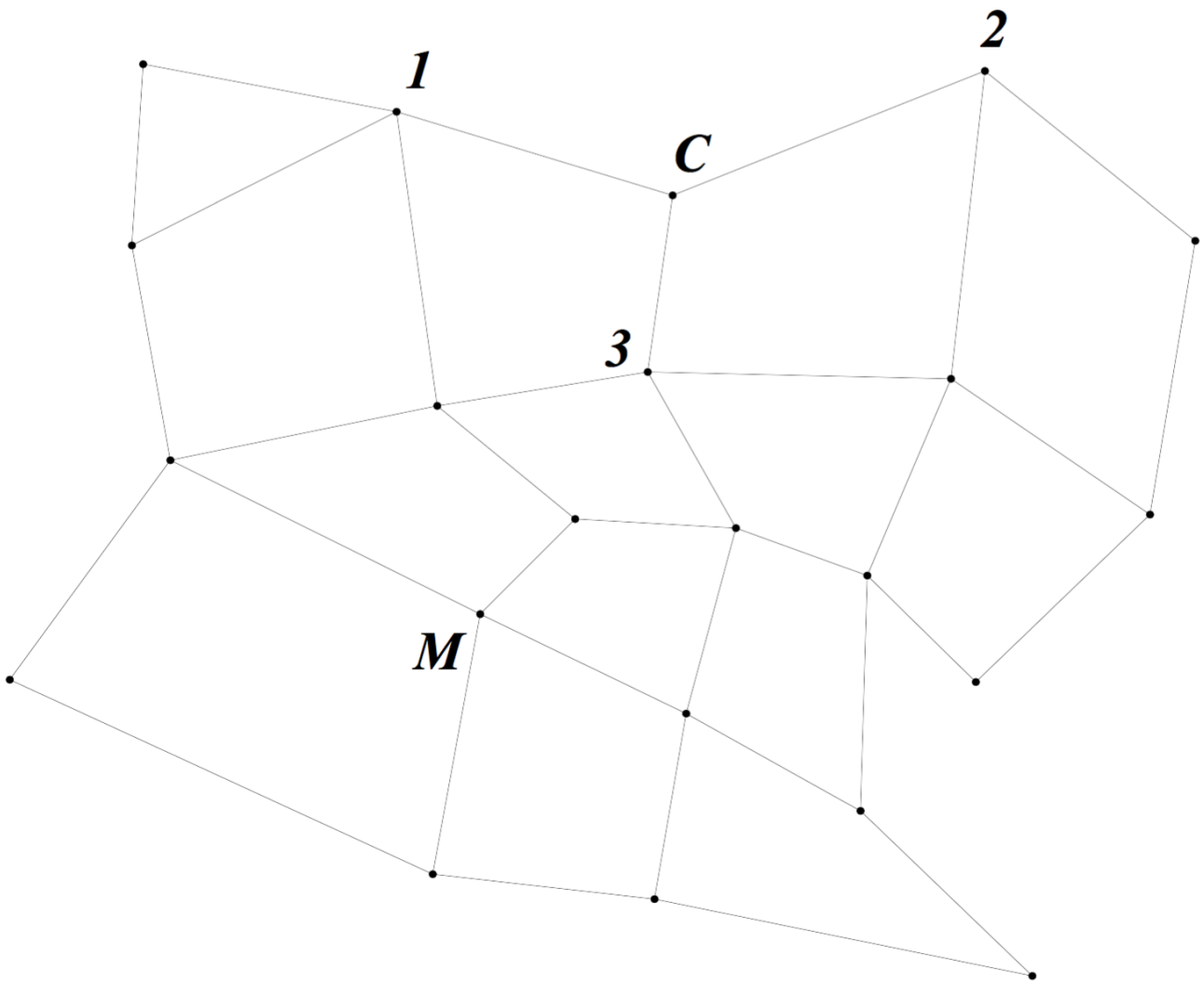


figure 5

figure 6

3 Cat and Mouse

8. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?