CHMC Advanced Group: Euler Characteristic and Planar Graphs

03/24/2018

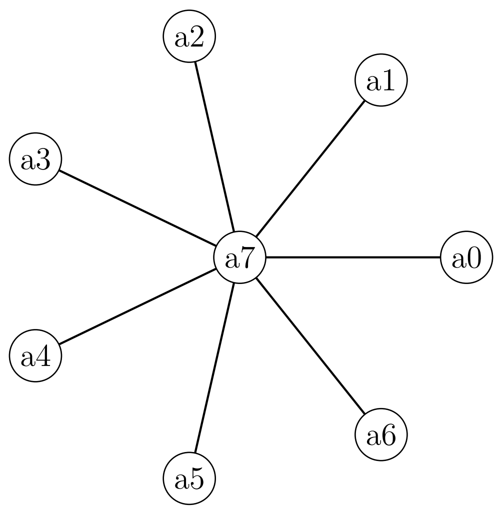
1 Introduction

This worksheet explores a property of graphs known as planarity, and determines when a graph can be drawn without overlapping edges.

2 The basics

A **simple graph** $G = (V, E)$ is a set V of distinct vertices and a set E of distinct edges with certain restrictions. More precisely, we will look at **finite graphs** for which V and E are finite. Associated to each edge $e \in E$ is an unordered pair of distinct vertices $\{v_i, v_j\}$; we do not care about the direction the edge travels, just the vertices it connects. We also will only consider graphs where any pair of vertices is connected by at most one edge.

To each vertex $v \in V$, we denote by $d(v)$ the **degree** of that vertex, which just counts the number of edges attached to the vertex. Since edges are given by distinct pairs of vertices, there are no loops in the graph, so we do not double count an edge when calculating $d(v)$.

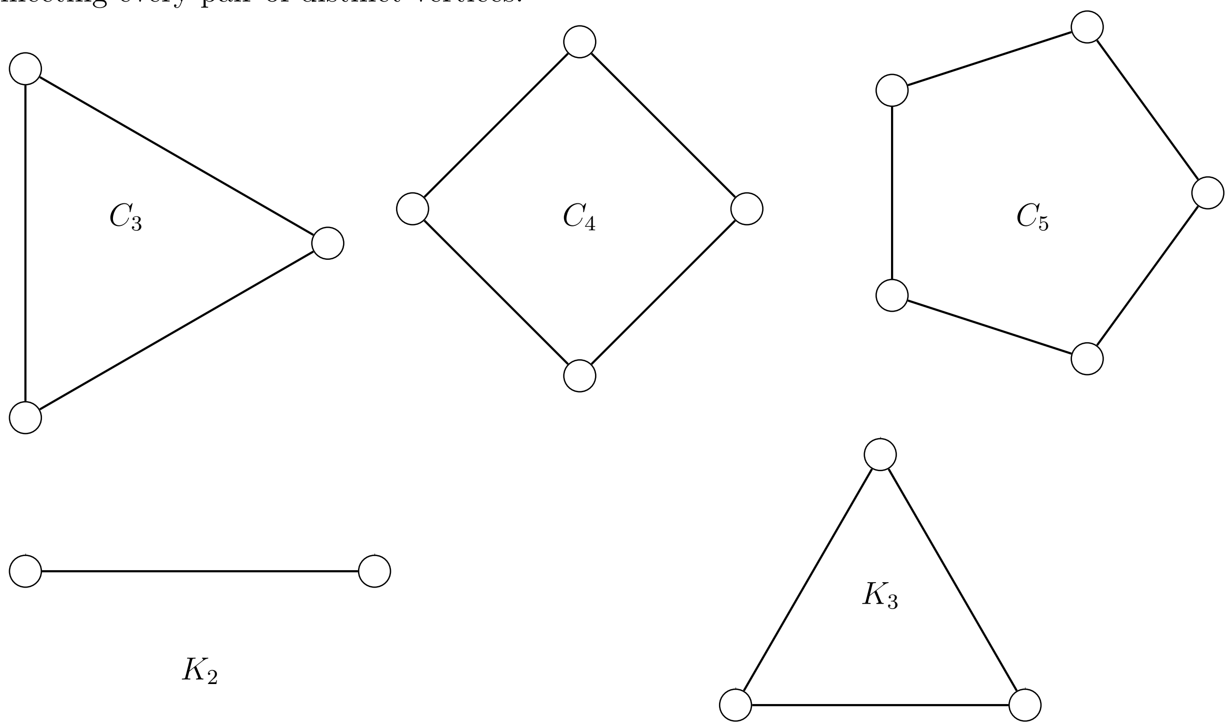


Exercise 2.1 For the graph pictured above, what is the degree of the central vertex? What is the degree of the outer vertices?

A **path** P is an ordered list of vertices $v_1 v_2 \cdots v_k$, where each pair of consecutive vertices are connected by an edge, i.e., the path consists of the edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$. Since edges are uniquely determined by a pair of vertices, there is only one such edge that can connect a pair of vertices. The **length** of a path is equal to the number of edges in the path. In the graph above, a path from the vertex labelled a_0 and the vertex labelled a_4 is $\{a_0, a_7\}, \{a_7, a_4\}$. This is a path of length 2. A graph is **connected** if there is a path between any two vertices in the graph. The graph above is in fact a connected graph. We will focus our attention to connected graphs in this worksheet.

Exercise 2.2 Draw a graph with 5 edges, with 7 edges, and with 10 edges. Now, for a given graph, add up $d(v)$ for each vertex in the graph. How does this relate to the number of edges in the graph? For a graph with m edges, conjecture what the sum of the degrees of vertices in the graph is equal to.

There are a few special types of graphs. A **cycle** C_n is a path where the first vertex is equal to the last vertex and is of length $n \geq 3$. A **tree** is a graph in which no cycle exists. The **complete graph on n vertices** K_n is the graph consisting of n vertices and an edge connecting every pair of distinct vertices.



Exercise 2.3 Draw a copy of K_4, K_3, K_6. How many edges does each complete graph have? Can you relate it to the number of vertices? How many edges does K_n have for $n \geq 1$?

Exercise 2.4 Draw a tree with 3 vertices, a tree with 4 vertices, and a tree with 5 vertices. How many edges does each tree have? How many edges does a tree with n vertices have? What happens if you add one more edge (think about the type of paths that exist in the tree)?

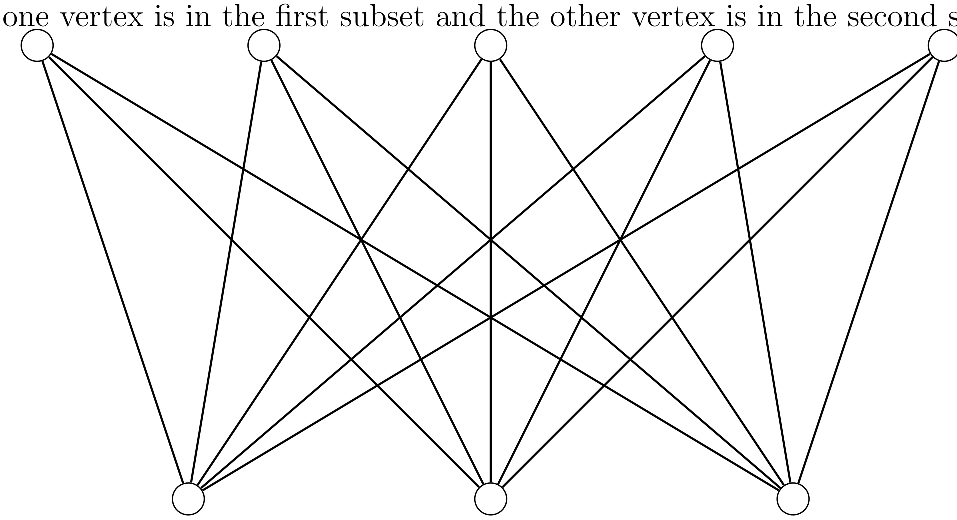
We call a graph **planar** if it can be drawn in the plane such that no two distinct edges cross. Here, edges need not be straight but instead can be represented as arcs. If a graph cannot be drawn without distinct edges overlapping, we call it **non-planar**.

Exercise 2.5 Of the graphs you drew in the previous two exercises, which ones are planar? Which ones are non-planar? For each graph, count the number vertices and the number of edges. Subtract the number of edges from the number of vertices. What do you observe?

A graph is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

Exercise 2.6 Draw the graphs C_3, C_4, C_5 and C_6. Are any of these bipartite? Conjecture when C_n is a bipartite graph and when it is not.

The **complete bipartite graph** $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.



$K_{3,5}$

Exercise 2.7 Draw $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$. How many edges does each graph have? Conjecture on the number of edges in the graph $K_{m,n}$. Are any of these graphs planar? If so, which ones?

Given a graph $G = (V, E)$, a **subgraph** G' of G is defined by $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. That is, a subgraph G' consists of a collection of vertices V' from the original set of vertices V in the graph, and a collection of edges E' from the original collection of edges E of the graph. We may think of a subgraph G' of G as a graph obtained from G by possibly deleting vertices and edges from G.

Exercise 2.8 In the complete graphs K_3, K_4, find different subgraphs that are trees and are cycles. Notice, if $m \leq n$, then K_m is a subgraph of K_n. How many subgraphs of K_n can you find that look like K_m? Try this first for K_2 in K_4, K_2 in K_5, K_3 in K_4 and K_3 in K_5. Once you have an idea, conjecture on the number of subgraphs of K_n that look like K_m.

We can create a new graph from a given graph $G = (V, E)$ by **contracting** and edge. Let v_i, v_j be distinct vertices with edge $\{v_i, v_j\}$ connecting them. Contracting the edge $\{v_i, v_j\}$

means we shrink the edge so that the vertices v_i and v_j are the same. All edges that were connected to either v_i or v_j are now connected to this new vertex. A graph $X = (V_X, E_X)$ constructed by contracting edges in $G' = (V', E')$, a subgraph of $G = (V, E)$, and deleting any loops or parallel edges is called a **minor** of the graph G. A special case of this is called a **topological minor**, which is constructed as follows. Pick a vertex set $V' \subset V$ and consider certain paths in G connecting vertices in V' as edges. Contract all other edges and view the paths as edges. This new graph is a topological minor of G.

Exercise 2.9 Draw a tree with 6 vertices. Contract 3 edges. What type of graph is left? Consider the cycle C_7 and contract 3 edges. What is the resulting graph? Conjecture what type of graph results from contracting edges in a tree and what type of graph results from contracting edges in a cycle.

Exercise 2.10 Consider the complete graph K_n and contract an edge. What type of graph results from this edge contraction? Try this for various K_n and form a conjecture on what edge contractions result in.

Exercise 2.11 For K_5 construct a minor on 3 vertices. Do the same for K_4. Now, draw a graph on 10 vertices and construct a topological minor on 4 vertices.

3 Plane Graphs

In this section we explore plane graphs more thoroughly and some properties they possess.

A more precise definition of a plane graph $G = (V < E)$ is as follows:

i) $V \subset \mathbb{R}^2$;
ii) every edge is an arc between two vertices;
iii) different edges have different endpoints;
iv) the interior of an edge contains no vertex and no point of any other edge.

For every plane graph G, the set $\mathbb{R}^2 \backslash G$ is open; its regions are called the **faces** of G. The outermost face - the unbounded face - is called the **outer face** of G and the other faces are called the **inner face**.

A **plane triangulation** is a graph G in the plane for which every face is bounded by a triangle.

Exercise 3.1 Draw a tree on 5 vertices. Add edges to this graph until every face, including the outer face, is bounded by a triangle. How many edges doe this graph have? Now, do the same with C_7, the cycle on 7 vertices. How many edges does this have in total? Can you add an edge to either of these graphs and remain planar?

It turns out that a graph that is a plane triangulation is maximal in the number of edges a planar graph can possess, that is, any addition of an edge to the graph would result in the graph losing its planar property.

Exercise 3.2 Draw a tree on 5, 6, and 7 vertices. Which of these, if any, are planar? Do the same with C_3, C_4, C_5. Again, which of these is planar? Finally, draw K_3, K_4, K_5 and determine which are planar. For those which are planar, verify that $v - e + f = 2$, where v denotes the number of vertices, e the number of edges, and f the number of faces.

In fact, any planar graph satisfies $v - e + f = 2$. For a test of planarity, consider the following.

Given a plane triangulation on n vertices, notice that every edge lies on the boundary of two faces. Furthermore, every face is bounded by a triangle. Hence, to each face we count three edges, but these are counted twice, so $3f = 2e$. Thus, $f = \frac{2e}{3}$ and plugging into $v - e + f = 2$, we see $2 = v - e + \frac{2e}{3} = v - \frac{e}{3}$. Hence, $e = 3v - 6$. Since, plane triangulations are maximal in the number of edges they possess, this gives a bound on the number of edges a planar graph on n vertices can have, that is, $\leq 3n - 6$.

Exercise 3.3

Verify that the planar graphs you considered in the previous exercise satisfy this inequality and the graphs which are not planar, fail this. Look at the complete graphs K_3, K_4, and K_5. Which of these are planar? Conjecture which graphs K_n are planar using the bound $e \leq 3v - 6$.

For the complete bipartite graph $K_{m,n}$ every edge is on a cycle of length 4 and each cycle of length 4 bounds a face. Hence, every face counts 4 edges and each of these edges is counted twice, that is $2f = e$. Thus, for a bipartite graph, $2 = v - e + f = v - \frac{3e}{2}$. For a bipartite graph which is maximal, this gives an upper bound, so $3e \leq 2v - 4$.

Exercise 3.4 Consider the complete bipartite graphs $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$. Which of these is planar? Conjecture on which $K_{m,n}$ are planar?

It is a well known theorem that a graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor or as a topological minor.

4 Higher genus

We now consider surfaces, such as the sphere or the torus. The torus is doughnut shaped, although it does not contain the filling, i.e., it is hollow. We can triangulate the sphere or torus by drawing on each surface a graph where every face is bounded by a triangle. If we doe this, we can define the **Euler characteristic** of the surface S by $\chi(S) = v - e + f$.

Exercise 4.1 Draw a sphere and a torus. Now, on each of these, form a triangulation. What is the Euler characteristic of each of these surfaces?

We define the **genus** of a surfaces as the integer g such that $2 - 2g = \chi(S)$. Hence, a sphere is a genus 0 surface and the torus is a genus 1 surface. There are higher genus surfaces that we can construct. The torus can be thought of as a sphere in which two small disks were removed and with a tube glued to each circle. In the same way, we can continue this process to create higher genus surfaces.

By adapting the argument from the previous section, a graph in a surface S with Euler characteristic $\chi(S)$ has a bound on the number of edges for a given number of vertices, namely $e \leq 6v - 2\chi(S)$. Also, since $2 - 2g = \chi(S)$, this gives $e \leq 6v - 4 + 4g$ for a genus g surface.

Similarly, for bipartite graphs, the edge bound is $3e \leq 2v - 2\chi(S)$. This gives $3e \leq 2v - 4 + 4g$ for a genus g surface.

Exercise 4.2 On what genus surface can K_5 be drawn such that no distinct edges cross? Try to realize this with a drawing. Do the same for $K_{3,3}$.