CHMC Advanced Group: Quaternions

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1 Introduction

This worksheet is an introduction to **quaternions**. Quaternions, roughly speaking, are a four-dimensional generalization of complex numbers.

The second section is a quick review of some assorted properties of complex numbers. The third section introduces quaternions, and explores some basic properties they possess. The fourth section explores two matrix representations of quaternions.

The final section is a brief introduction into the role quaternions play in regards to rotations. The idea is that, the real part of a quaternion encodes most of the information about the angle of rotation, while the other three components encode information about the axis of rotation.

2 Review of complex numbers

A **complex number** z is a number of the form $z = a + b\sqrt{-1}$, where a, b are real numbers. We also have the **complex conjugate** $\bar{z} = a - b\sqrt{-1}$. If we think of the complex number $a + b\sqrt{-1}$ as a point (a, b) in the plane, then the number $a^2 + b^2$ gives us the distance from (a, b) to the origin.

Exercise 2.1 Show that, if $z = a + b\sqrt{-1}$, then $a^2 + b^2 = z\bar{z}$.

Complex numbers have some very important *algebraic* properties: any equation of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ has exactly n complex solutions for x (not necessarily distinct). For example, the polynomial $x^2 + 1 = 0$ has no real solutions, but it does have two complex solutions: $x = \sqrt{-1}$ and $x = -\sqrt{-1}$ both satisfy $x^2 + 1 = 0$. Here, though, we're more interested in the *geometric* properties that complex numbers have.

Exercise 2.2 Consider the complex number $z = 1 + \sqrt{-1}$, and the corresponding point in the plane $\left(1, \sqrt{-1}\right)$. What is the number $z \cdot \sqrt{-1}$? What is the corresponding point in the plane?

The last exercise illustrates (by a specific example) that multiplication by $\sqrt{-1}$ corresponds to a rotation by 90°. The last section of this worksheet explores a generalization of this idea to 3-dimensions.

Finally, we recall a matrix formulation of complex numbers. Much of this material was set out in a previous CHMC worksheet, and the interested reader can explore more details there.

Exercise 2.3 Show that the matrix

$$[\sqrt{-1}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfies

$$[\sqrt{-1}]^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3 Quaternions

A **quaternion** q is a number of the form $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the following properties:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$
$$\mathbf{i}\,\mathbf{j} = \mathbf{k}, \mathbf{j}\,\mathbf{k} = \mathbf{i}, \text{and } \mathbf{k}\,\mathbf{i} = \mathbf{j}\,.$$

Exercise 3.1 Show, using the properties above, that $\mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$.

Exercise 3.2 Show, using the properties above, that $\mathbf{j}\,\mathbf{i} = -\mathbf{k}, \mathbf{k}\,\mathbf{j} = -\mathbf{i}$, and $\mathbf{i}\,\mathbf{k} = -\mathbf{j}$.

Exercise 3.3 What is $\mathbf{i}^{-1}$? In otherwords, what quaternion q must we multiply $\mathbf{i}$ by to get $q \cdot \mathbf{i} = 1$?

Exercise 3.4 Show that $\mathbf{i}^{-1}\,\mathbf{j}\,\mathbf{i} = \mathbf{j}^{-1}$.

Quaternions have a multiplicative structure that behaves a lot like the complex and real numbers. For example, to multiply $p = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ with $q = e + f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$, we distribute the terms and then recombine:

$$\begin{aligned}
pq =& (a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})(e + f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}) \\
=& ae + af\,\mathbf{i} + ag\,\mathbf{j} + ah\,\mathbf{k} + be\,\mathbf{i} + bf\,\mathbf{i}\,\mathbf{i} + bg\,\mathbf{i}\,\mathbf{j} + bh\,\mathbf{i}\,\mathbf{k} \\
&+ ce\,\mathbf{j} + cf\,\mathbf{j}\,\mathbf{i} + cg\,\mathbf{j}\,\mathbf{j} + ch\,\mathbf{j}\,\mathbf{k} + de\,\mathbf{k} + df\,\mathbf{k}\,\mathbf{i} + dg\,\mathbf{k}\,\mathbf{j} + dh\,\mathbf{k}\,\mathbf{k} \\
=& ae + af\,\mathbf{i} + ag\,\mathbf{j} + ah\,\mathbf{k} + be\,\mathbf{i} + bf(-1) + bg\,\mathbf{k} + bh(-\mathbf{j}) \\
&+ ce\,\mathbf{j} + cf(-\mathbf{k}) + cg(-1) + ch(\mathbf{i}) + de\,\mathbf{k} + df(\mathbf{j}) + dg(-\mathbf{i}) + dh(-1) \\
=& (ae - bf - cg - dh) + (af + be + ch - dg)\,\mathbf{i} \\
&+ (ag - bh + ce + df)\,\mathbf{j} + (ah + bg - cf + de)\,\mathbf{k}\,.
\end{aligned}$$

It is not recommended at all that this formula is memorized. Rather, whenever two quaternions need to be multiplied go ahead and distribute the terms and do everything from scratch. Many of the quaternions we'll be working with in the last section will only have three non-zero components, rather than all four.

Exercise 3.5 Let $p = 1 + 3\,\mathbf{i} + \mathbf{k}$, $q = 2\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$, and $r = 4 + 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. Compute pq, pr and rp.

Exercise 3.6 In general, suppose p and q are two quaternions. Is it true that $pq = qp$? Why or why not? If not, give an explicit example.

Exercise 3.7 If p, q, and r are three quaternions, is it true that $(pq)r = p(qr)$? Here, the parenthesis indicate which quaternions are to be multiplied together first. If so, show this.

Exercise 3.8 Complex numbers are a special case of quaternions: why?

We define the **magnitude** or **norm** of a quaternion $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ to be the number

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

A useful definition, similar to that in the complex case, is that of a **conjugate** quaternion: if $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, then the conjugate of q is the quaternion

$$\bar{q} = a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}\,.$$

Exercise 3.9 Show that $\|q\| = \sqrt{q\bar{q}}$.

The next exercise establishes a useful property that the quaternion norm satisfies, much like the ordinary absolute value does for real numbers.

Exercise 3.10 Show that if q and r are two quaternions, then $\|qr\| = \|q\|\|r\|$.

Exercise 3.11 Is it possible for $\|q\| = 0$, but $q \neq 0$? Why or why not? Recall that $q = 0$ means $a = b = c = d = 0$ for $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$.

In the last section we'll use the inverse of arbitrary quaternions, so let's verify that they take a particularly convenient form:

Exercise 3.12 Show that if $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, then $q^{-1} = \frac{\bar{q}}{\|q\|^2} = \frac{a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}}{\|q\|^2}$.

Exercise 3.13 What is r^{-1}, where $r = 4 + 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$?

4 Representations of quaternions

In the first section, we noticed that the matrix $[\sqrt{-1}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies the property that

$$[\sqrt{-1}]^2 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is the matrix analogue of the property that $\sqrt{-1}$ has. In this section we'll explore two representations of quaternions with matrices.

The first representation will involve matrices whose entries are complex numbers. Consider the two matrices

$$[\mathbf{i}]_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad [\mathbf{j}]_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again, the subscript of 1 indicates that this is the first of two representations we'll consider in this section.

Exercise 4.1 Verify that, indeed, $([\mathbf{i}]_1)^2 = -I$ and $([\mathbf{j}]_1)^2 = -I$.

Exercise 4.2 Using the properties that $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ must satisfy from before, what must the matrix $[\mathbf{k}]_1$ be? Verify that, for this matrix, we have $([\mathbf{k}]_1)^2 = -I$.

Exercise 4.3 Recall that $\mathbf{i}^{-1} = -\mathbf{i}$. What is $([\mathbf{i}]_1)^{-1}$? Use this, and the fact that $\mathbf{i}^{-1}\,\mathbf{j}\,\mathbf{i} = \mathbf{j}^{-1}$ to compute the matrix $([\mathbf{j}]_1)^{-1}$.

Now we look at writing an arbitrary quaternion $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ as a matrix. The idea is to replace each of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ with their matrix equivalents. What do we do about the a term though? The most likely candidate is to replace the constant 1 with the matrix I, i.e. $[1]_1 = I$. In this way, we get

$$[a]_1 = a[1]_1 = aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Thus, as a matrix, we have

$$[a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}]_1 = a[1]_1 + b[\mathbf{i}]_1 + c[\mathbf{j}]_1 + d[\mathbf{k}]_1.$$

The terms of the right hand side can be added together to obtain a single 2×2 matrix, giving the matrix representation of our quaternion.

Exercise 4.4 Let $q = 2\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ and $r = 4 + 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. What are the mmatrices $[q]_1$ and $[r]_1$? What is the matrix product $[q]_1[r]_1$? Does this agree with your findings from exercise 3.5?

The second representation we'll consider will not involve complex numbers as matrix elements, but will involve 4×4 matrices with real entries. In a manner similar to before, let's set

$$[\mathbf{i}]_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad [\mathbf{j}]_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Some of the intuition for why these matrices are chosen comes from the fact that $bfi \cdot 1 = 1, \mathbf{i} \cdot \mathbf{i} = -1, \mathbf{i} \cdot \mathbf{j} = \mathbf{k}$, and $\mathbf{i} \cdot \mathbf{k} = -\mathbf{j}$. The matrix $[\mathbf{i}]_2$ keeps track of what happens when we multiply each of these "basic" quaternions $1, \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ by the quaternion $\mathbf{i}$. For example, the third column has a -1 in the fourth row, because multiplying the third "basic" quaternion $\mathbf{j}$ on the left by the quaternion $\mathbf{i}$ results in negative the fourth "basic" quaternion $\mathbf{k}$.

Exercise 4.5 With the same reasoning as in the last paragraph, why would we expect $[\mathbf{j}]_2$ to have the form that it does?

Exercise 4.6 Verify that

$$([\mathbf{i}]_2)^2 = ([\mathbf{j}])^2 = -I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Exercise 4.7 Since $\mathbf{i}\,\mathbf{j} = \mathbf{k}$, what must $[\mathbf{k}]_2$ be?

Exercise 4.8 Let $q = 2\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ and $r = 4 + 3\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. What are the matrices $[q]_2$ and $[r]_2$? Check that the matrix product $[q]_2[r]_2$ agrees with what you found in exercises 3.5 and 4.4.

5 Quaternions as rotations

We call a quaternion **pure** if it doesn't have a real part. In other words, a pure quaternion is of the form $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{j}$. The goal of this section is to explore the fact that, if $q = \cos\theta + \sin\theta r_1\,\mathbf{i} + \sin\theta r_2\,\mathbf{j} + \sin\theta r_3\,\mathbf{k}$ is a pure quaternion and $\|q\| = 1$, then a rotation in 3-dimensions of a point (a, b, c) about the axis of rotation (r_1, r_2, r_3) by an angle of 2θ corresponds to the quaternionic multiplication $q \cdot (a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) \cdot q^{-1}$.

Exercise 5.1 Show that if $r_1^2 + r_2^2 + r_3^2 = 1$, then $q = \cos\theta + \sin\theta r_1\,\mathbf{i} + \sin\theta r_2\,\mathbf{j} + \sin\theta r_3\,\mathbf{k}$ satisfies $\|q\| = 1$.

To simplify the notation, if q is a quaternion then we'll write $R_q(p)$ for the operation qpq^{-1}. With this framework, the last equation above takes the form $R_{\cos\theta + \sin\theta r_1\,\mathbf{i} + \sin\theta r_2\,\mathbf{j} + \sin\theta r_3\,\mathbf{k}}(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$, i.e. more succinctly as $R_q(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$.

Exercise 5.2 Verify that we have $R_q(R_p(r)) = R_{qp}(r)$, for any quaternions p, q, r.

Exercise 5.3 Show that if a is any real number, and p, q are two quaternions, then $R_{ap}(q) = R_p(q)$.

Exercise 5.4 Show that if $R_q(r) = p$, then $R_{q^{-1}}(p) = r$.

Exercise 5.5 Show that $\|R_q(r)\| = \|r\|$ for any pure quaternion r.

The last three exercises illustrate the fact that this rotation operator obeys composition laws, i.e. that if you rotate by the axis and angle combination given by p, and then by q, is the same as rotating by the axis and angle pair given by the quaternion product qp; and that if you scale the quaternion of rotation, then you get the same rotation operator; the inverse of this rotation operator is rotation by the inverse of the quaternion.

Let's explore some examples. Let $\theta = 45°$, so that $\cos\theta = 0$ and $\sin\theta = 1$. The corresponding rotation quaternion, for axis of rotation (r_1, r_2, r_3), is $q = r_1\,\mathbf{i} + r_2\,\mathbf{j} + r_3\,\mathbf{k}$.

Exercise 5.6 Let the axis of rotation be $(0, 0, 1)$, so that $q = \mathbf{k}$. What is the angle of rotation for this quaternion? If $r = \mathbf{i} + \mathbf{j} + \mathbf{k}$, what is the quaternion product $R_q(r)$? What point in 3-dimensions does the quaternion r correspond to? What about the quaternion $R_q(r)$? Does this quaternionic multiplication correspond to a rotation of the point $(1, 1, 0)$ about the axis $(0, 0, 1)$?

Exercise 5.7 Same exercise as above, this time with the axis of rotation $(1, 1, 1)$ and the specific point $(1, 2, 3)$.