# Further Topics in Mass Point Geometry

## 1 Review: Mass Point Geometry

Let's recall some of the basic definitions and results about mass point geometry which we introduced last week. Our approach is modeled on that of [1] its subsequent adaptation in [2].

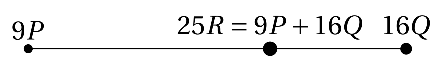
**Definition 1.1.** Let $P$ be a point, and $m > 0$ a real number. Then a *mass point* is an ordered pair $(m, P)$. Two mass points $(m, P)$ and $(n, Q)$ are defined to be equal if and only if $m = n$ and $P = Q$. *Notation:* henceforth, we shall write $mP$ to denote the mass point $(m, P)$. We shall often simply write "$P$" as shorthand for the mass point $1P = (1, P)$, too, where the context is clear that this represents a mass point and not simply a point.

**Definition 1.2.** Let $mP$ and $nQ$ be two mass points. The *mass point sum* of $mP$ and $nQ$, denoted $mP + nQ$, is defined as follows:

- If $P = Q$, then $mP + nQ := (m+n)P$.
- If $P \neq Q$, then $mP + nQ$ is defined to be mass point of mass $m+n$ at the unique point $R$ on the line segment $\overline{PQ}$ that lies $n/(m+n)$ of the distance $PQ$ from $P$ to $Q$. That is,

$$\frac{PR}{RQ} = \frac{n}{m}.$$

- If $mP$ is a mass point and $a > 0$, then we define $a(mP) := (am)P$.




Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

**Proposition 1.3.** *Let $\ell O$, $mP$, and $nQ$ be mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative: $mP + nQ = nQ + mP$.*
- *Mass point addition is associative: $\ell O + (mP + nQ) = (\ell O + mP) + nQ$*
- *Mass point addition is distributive: $a(mP + nQ) = amP + anQ$.*

Proving both commutativity and distributivity is relatively straightforward. The associativity of mass point addition, however, is more complicated.

The following are strategies for using mass points:

1.1. Assume that $\overline{PQ}$ is a line segment containing the point $R$. Then if we know $PR/RQ = n/m$, assign masses $m$ and $n$ to $P$ and $Q$ respectively so that $mP + nQ = (m+n)R$. Further, we can scale this by a positive constant $k$ to have $k(m+n)R = kmP + kmQ$, as well.

Recall that the numerator in this ratio is assigned to $Q$ and the denominator is assigned to $P$, so that the *larger* mass is assigned to which of $P$ or $Q$ is *closer* to $R$.

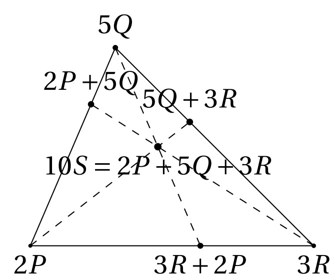
1.2. Given mass points $mP$ and $nQ$, use mass point operations to determine the location of $R$ on $\overline{PQ}$ such that $mP + nQ = (m+n)R$.

As an example, say we establish that $mP + nQ = tR + uS$. Then by the definition of mass point, $mP + nQ$ must lie on $\overline{PQ}$, and $tr + uS$ must lie on $\overline{RS}$. If these mass points are the same, then the underlying point at which they agree must be where the two line segments intersect.

1.3. We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.

For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP + nP$.

As a reminder of how mass point operations are relevant for cevians in triangles, if $P$, $Q$, and $R$ are the vertices of a triangle, then the mass point sum $2P + 5Q + 3R$ is indicated below:




(As a reminder: a *cevian* of a triangle $\triangle ABC$ is a line segment, one of whose endpoints is a vertex of $\triangle ABC$, and where the other endpoint is any point of the opposite side, excluding the endpoints.)

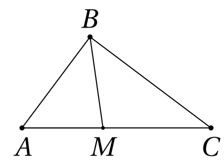
## 2 Geometric Background: The Law of Sines, Angle Bisectors, and Areas

Let us review some preliminary results which will be useful in combination with techniques from mass point geometry. These can be proven, but they do not require mass point geometry.

*Notation:.* If $\triangle ABC$ is a triangle, then unless otherwise indicated, we let $A$, $B$, and $C$ denote angles $\angle BAC$, $\angle ABC$, and $\angle ACB$, respectively. Further, we let $a$, $b$, and $c$ denote the lengths $BC$, $AC$, and $AB$, respectively.

**Proposition 2.1.** *Let $\triangle ABC$ be a triangle. If $M$ lies on $\overline{AC}$, then $\overline{BM}$ bisects $\angle ABC$ if and only if*

$$\frac{c}{a} = \frac{AM}{MC}.$$



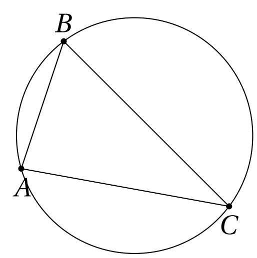

**Proposition 2.2.** *Let $\triangle ABC$ be a triangle, with $M$ a point on $\overline{AC}$. Then*

$$\frac{\text{Area}(\triangle ABM)}{\text{Area}(\triangle BMC)} = \frac{AM}{MC}.$$

*That is, the ratio into which cevian $\overline{BM}$ divides the length of $\overline{AC}$ is the same as the ratio into which it divdes the area of $\triangle ABC$.*

**Theorem 2.3** (The Law of Sines)**.** *Let $\triangle ABC$ be a triangle, where $R$ is the radius of it circumcircle. Then*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

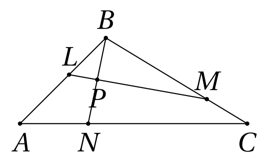



*Remark.* For earlier exercises with mass point geometry, we assigned masses when we had information about the ratios of lengths of sides. If we have information about angles—or, equivalently, about their trigonometric values—then we may use that information to assign masses to points in such a way that relevant mass point sums balance at the endpoint of a cevian. See Exercises #3.3 and #3.4 for examples.

## 3 Exercises

We begin by restating what was Exercise 6.1 from last week's worksheet, this time with a solution:

3.1. Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point $P$, as below. If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios $LP/PM$ and $BP/PN$.



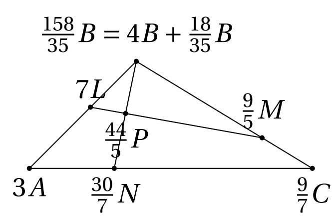

*Solution:* We'll use Strategy #1.3 by "splitting" a mass point at $B$. Begin by assigning mass 3 to $A$, which after considering $AL/LB$ would assign mass 4 to $B$. Next, we balance $\overline{AC}$ at $N$, assigning mass $3 \cdot \frac{3}{7} = \frac{9}{7}$ to $C$. Having formed the mass points $3A$, $4B$, and $\frac{9}{7}C$, though, we see a possible inconsistency: since $BM/MC = 5/2 \neq \frac{9}{7}/4 = \frac{9}{28}$, the point $M$ does not lie at the point for the mass point sum $4B + \frac{9}{7}C$. How to resolve this dilemma?

To balance $\overline{BC}$ at $M$, we assign a second, *distinct* mass $\frac{9}{7} \cdot \frac{MC}{BC} = \frac{9}{7} \cdot \frac{2}{5} = \frac{18}{35}$ to $B$ so that $\frac{18}{35}B + \frac{9}{7}C = \frac{63}{35}M = \frac{9}{5}M$. We have the two mass points $4B$ and $\frac{18}{35}B$ at $B$, and we emphasize this "splitting" by simply writing it as $(4 + \frac{18}{35})B$.

The system has total mass $3 + (4 + \frac{18}{35}) + \frac{9}{7} = \frac{44}{5}$. To recall, we have the following mass points assigned: $3A$, $4B$, $\frac{18}{35}B$, $\frac{9}{7}C$. Further, we have that

$$\begin{aligned} 7L &= 3A + 4B \\ \frac{9}{5}M &= \frac{18}{35}B + \frac{9}{7}C \\ \frac{30}{7}N &= 3A + \frac{9}{7}C. \end{aligned}$$

See the revised diagram below:




Let us consider the mass point sums $7L + \frac{9}{5}M$ and $(4B + \frac{18}{35}B) + \frac{30}{7}N$. We have, by Proposition 1.3, that

$$\begin{aligned} 7L + \frac{9}{5}M &= (3A + 4B) + \left(\frac{18}{35}B + \frac{9}{7}C\right) \\ &= 3A + 4B + \frac{18}{35}B + \frac{9}{7}C \\ &= \left(4B + \frac{18}{35}B\right) + \left(3A + \frac{9}{7}C\right) \\ \implies 7L + \frac{9}{5}M &= \left(4 + \frac{18}{35}\right)B + \frac{30}{7}N. \end{aligned}$$

By definition, the mass point sum $7L+\frac{9}{5}M$ lies on $\overline{LM}$, and the mass point sum $(4+\frac{18}{35})B+\frac{30}{7}N$ lies on $\overline{BN}$. We showed above that these mass points are identical, so we must have that the mass point lies on a point common to both $\overline{LM}$ and $\overline{BN}$. This mass point thus lies at $P$, the intersection of the two line segments.

Therefore, we have $7L+\frac{9}{5}M=(7+\frac{9}{5})P=\frac{44}{5}P$, meaning $\underline{LP/PM=\frac{9}{5}/7=\frac{9}{35}}$. Furthermore, $(4+\frac{18}{35})B+\frac{30}{7}N=\frac{44}{5}P$, whence $\underline{BP/PN=(\frac{30}{7})/(\frac{158}{35})=\frac{75}{79}}$. □

3.2. *Varignon's Theorem:* Let $A$, $B$, $C$, and $D$ be the vertices in the plane of a (nondegenerate) quadrilateral. Let $K$, $L$, $M$, and $N$ be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that $KLMN$ is a parallelogram. *Note:* we are not assuming that $ABCD$ is a convex quadrilateral.

3.3. Let $\triangle ABC$ be a triangle with cevian $\overline{BM}$ bisecting $\angle ABC$.

(a) Show that
$$\frac{AM}{MC}=\frac{\sin C}{\sin A}.$$
Equivalently, show that $AM\sin A=MC\sin C$.

(b) Assume $\sin A=3/5$ and $\sin C=7/25$. Consider the median $\overline{AN}$, which bisects $\overline{BC}$, and let $P$ denote the intersection of $\overline{BM}$ and $\overline{AN}$. Compute $AP/PN$ and $BP/PM$.

3.4. Let $\triangle ABC$ be a triangle. Show that the angle bisectors of $\triangle ABC$ are concurrent. That is, show that the angle bisectors intersect in a single point.

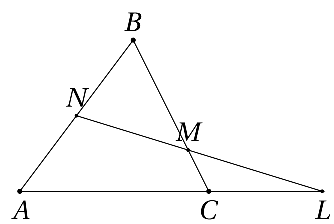
3.5. Let $\triangle ABC$ be an acute triangle. Prove that the altitudes of $\triangle ABC$ are concurrent.

3.6. Let $\triangle ABC$ be a triangle with cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$. Then the cevians are concurrent if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$

*Remark.* Note that we could have solved Exercises #3.4 and #3.5 by first proving Ceva's Theorem, then showing that the product given above equals 1 in each case.

3.7. *Menelaus' Theorem:* Let $\triangle ABC$ be a triangle, $L$ a point on the line given by $\overline{AC}$ as shown below, $N$ any point in $\overline{AB}$, and $M$ any point on $\overline{BC}$.




Then $L$, $M$, and $N$ are collinear if and only if

$$\frac{AN}{NB} \cdot \frac{BM}{MC} \cdot \frac{CL}{LA} = 1.$$

3.8. In $\triangle ABC$, if cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at $P$, then

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

## 4 Exercises from Mathematics Competitions

4.1. (**from ARML 1989**) In $\triangle ABC$, angle bisectors $\overline{AD}$ and $\overline{BE}$ intersect at $P$. If the sides of the triangle are $a = 3$, $b = 5$ and $c = 7$, with $BP = x$ and $PE = y$, then compute the ratio $x/y$.

4.2. (**from AHSME 1975**) In $\triangle ABC$, $M$ is the midpoint of side $\overline{BC}$, $AB = 12$, and $AC = 16$. Points $E$ and $F$ are taken on $\overline{AC}$ and $\overline{AB}$, respectively, and lines $\overline{EF}$ and $\overline{AM}$ intersect at $G$. If $AE = 2AF$, find $EG/GF$.

4.3. (**from ARML 1992**) In $\triangle ABC$, points $D$ and $E$ are on $\overline{AB}$ and $\overline{AC}$, respectively. The angle bisector of $\angle A$ intersetcs $\overline{DE}$ at $F$ and $\overline{BC}$ at $T$. If $AD = 1$, $DB = 3$, $AE = 2$, and $EC = 4$, compute the ratio $AF/AT$.

4.4. (**from AIME 1988**) Let $P$ be an interior point of $\triangle ABC$, and extend lines from the vertices through $P$ to the opposite sides. Let $AP = a$, $BP = 5$, $CP = c$, and let the extensions from $P$ to the opposite sides all have length $D$. If $a + b + c = 43$ and $d = 3$, then find $abc$.

## References

[1] Tom Rike, *Mass point geometry*, http://mathcircle.berkeley.edu/sites/default/files/archivedocs/2007_2008/lectures/0708lecturespdf/MassPointsBMC07.pdf, 2007, online: retrieved December 9, 2016.

[2] Zvezdelina Stankova, Tom Rike, and editors, *A decade of the Berkeley Math Circle: The American experience*, vol. I, Mathematical Sciences Research Institute and The American Mathematical Society, Providence, Rhode Island, USA, 2008.