An Introduction to Mass Point Geometry

1 Introduction: Mass Point Geometry

This week, we shall explore *mass point geometry*. Motivation comes from physics: we generalize the notion of the center of mass. Mass point geometry techniques can be powerful, offering much simpler ways to solve a number of geometric problems, both in the plane and higher-dimensionally. Our approach is modeled on that of [1] its subsequent adaptation in [2].

2 Preliminaries: Definitions and Notation

We begin with some notation and definitions:

Definition 2.1. Let P be a point, and $m > 0$ a real number. Then a *mass point* is an ordered pair (m, P). Two mass points (m, P) and (n, Q) are defined to be equal if and only if $m = n$ and $P = Q$. *Notation:* henceforth, we shall write mP to denote the mass point (m, P). We shall often simply write "P" as shorthand for the mass point $1P = (1, P)$, too, where the context is clear that this represents a mass point and not simply a point.

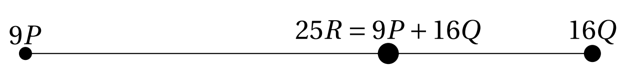
Definition 2.2. Let mP and nQ be two mass points. The *mass point sum* of mP and nQ, denoted $mP + nQ$, is defined as follows:

- If $P = Q$, then $mP + nQ := (m+n)P$.
- If $P \neq Q$, then $mP + nQ$ is defined to be mass point of mass $m + n$ at the unique point R on the line segment $\overline{PQ}$ that lies $n/(m+n)$ of the distance PQ from P to Q. That is,

$$\frac{PR}{RQ} = \frac{n}{m}.$$

- If mP is a mass point and $a > 0$, then we define $a(mP) := (am)P$.

Intuitively, the point R is the center of mass for a discrete system of masses with mass m at point P and mass n at point Q. If you prefer, imagine a balance beam with the given masses at the respective point; the balancing point is at R, and the entire system has overall mass $m + n$. See the example below:



Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

Proposition 2.3. *Let ℓO, mP, and nQ be mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative:* $mP + nQ = nQ + mP$.
- *Mass point addition is associative:* $\ell O + (mP + nQ) = (\ell O + mP) + nQ$
- *Mass point addition is distributive:* $a(mP + nQ) = amP + anQ$.

All these properties can (and should!) be justified, of course. For now, though, let's accept these provisionally and use the properties of mass points to solve certain exercises.

3 Basic Strategies with Mass Points

The following are strategies for using mass points:

3.1. Assume that $\overline{PQ}$ is a line segment containing the point R. Then if we know $PR/RQ = n/m$, assign masses m and n to P and Q respectively so that $mP + nQ = (m+n)R$. Further, we can scale this by a positive constant k to have $k(m+n)R = kmP + kmQ$, as well.

Recall that the numerator in this ratio is assigned to Q and the denominator is assigned to P, so that the *larger* mass is assigned to which of P or Q is *closer* to R.

3.2. Given mass points mP and nQ, use mass point operations to determine the location of R on $\overline{PQ}$ such that $mP + nQ = (m+n)R$.

3.3. We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.

For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP + nP$.

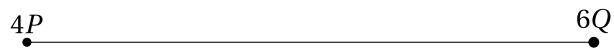
The second strategy in particular can be useful in showing, for example, that three line segments intersect in a common point. The third is invaluable when our desired assignment of mass points would otherwise be inconsistent with the given hypotheses.

4 Practice Exercises

4.1. Draw the following mass point sum $mP + nQ$:

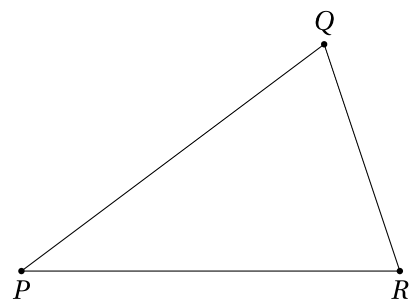
4.2. Draw the following mass point sum $mP + nQ$:

4.3. Draw the following mass point sum $mP + nQ$:

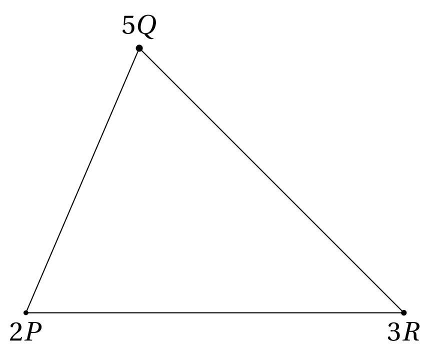


4.4. Draw the following mass point sum $mP + nQ$:

4.5. Draw the following mass point sum $1P + 1Q + 1R$:

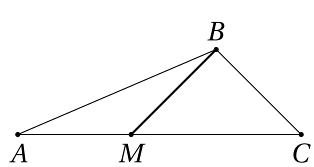


4.6. Draw the following mass point sum $2P + 5Q + 3R$:



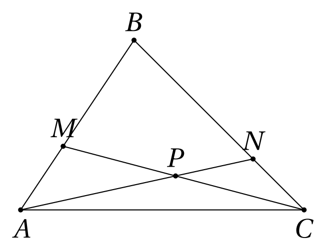
5 Cevians and Mass Points

Definition. Let $\triangle ABC$ be any triangle. A *cevian* is a line segment connecting one of the vertices of the triangle with any point (excluding the endpoints) of the opposite side. For example, $\overline{BM}$ below is a cevian of $\triangle ABC$.

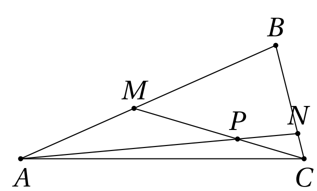


5.1. Consider a triangle $\triangle ABC$. A *median* of a triangle is a line segment whose endpoints are one of the triangle's vertices and the midpoint of the opposite side. Prove that the three medians of any triangle intersect in a common point, called the *centroid* of a triangle. Further, prove that the medians divide each other in the ratio 2-to-1, where the point of intersection lies farther from each vertex than from the opposite site.

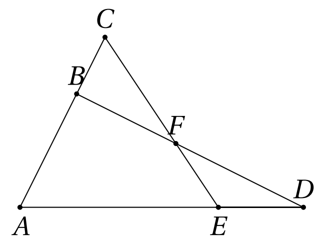
5.2. Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 3/5$ and $BN/NC = 7/3$. Compute the ratios AP/PN and CP/PM.



5.3. Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 4/5$ and $BN/NC = 7/2$. Compute the ratios AP/PN and CP/PM.

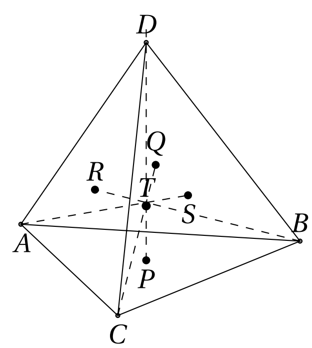


5.4. Consider the diagram below. If $AB/BC = 2$ and $AE/ED = 7/3$, then compute BF/FD and CF/FE.



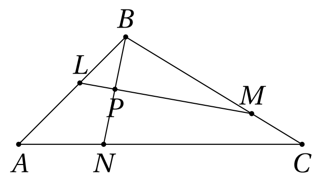
5.5. Consider a tetrahedron $ABCD$ in space. Let P, Q, R, and S be, respectively, the centroids of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$. (See Exercise #5.1 for the definition of the centroid of a triangle.) Prove that the line segments $\overline{AS}$, $\overline{BR}$, $\overline{CQ}$, and $\overline{DP}$ all intersect in a common point T. What are the ratios AT/TS, BT/TR, CT/TQ, and DT/TP?

Can you generalize this result to polyhedra in dimension 4 or higher?



6 Additional Exercises

6.1. Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point P, as below. If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios LP/PM and BP/PN.



6.2. Given mass points mP and nQ, how might you define the mass point difference $mP - nQ$? Under what conditions would $mP - nQ$ exist?

6.3. Say that mP and nQ are mass points, where in terms of Cartesian coordinates, $P := (x_1, y_1)$ and $Q := (x_2, y_2)$. What are the Cartesian coordinates of the point R, where $mP + nQ = (m+n)R$? If we are in three-dimensional space and the points have coordinates given by $P := (x_1, y_1, z_1)$, $Q := (x_2, y_2, z_2)$?

6.4. Earlier, we asked you to use Proposition 2.3 above without proving this. Here, prove each of these three claims.

References

[1] Tom Rike, *Mass point geometry*, http://mathcircle.berkeley.edu/sites/default/files/archivedocs/2007_2008/lectures/0708lecturespdf/MassPointsBMC07.pdf, 2007, online: retrieved December 9, 2016.

[2] Zvezdelina Stankova, Tom Rike, and editors, *A decade of the Berkeley Math Circle: The American experience*, vol. I, Mathematical Sciences Research Institute and The American Mathematical Society, Providence, Rhode Island, USA, 2008.