Probability, Part 1

1 Dice

1. Suppose you are rolling two dice. What is the probability that ...
   (a) the first die is an even number?
   (b) the second die is an even number?
   (c) both dice are even numbers?
   (d) you roll exactly one even number?
   (e) you roll at least one even number?

2. Now you are rolling three dice. What is the probability that ...
   (a) you roll all 6's?
   (b) you roll no 6's?
   (c) you roll at least one 6?

3. *Sum of Dice.* When two dice are rolled, there are 36 different outcomes, because $36 = 6 \times 6$. Each outcome is equally likely. Use this table to compute the various sums that can occur. A few cells are already filled in.

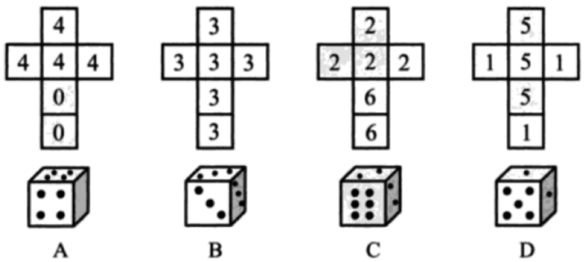
	1	2	3	4	5	6
1	2	3	4			
2		4				
3						
4						
5						
6						12

   (a) Verify that the probability of rolling two dice and getting a sum of 2 is 1/36
   (b) What is the probability of rolling two dice and getting a sum of 5?
   (c) What is the most likely sum, and what is its probability?

4. *Standard Dice.* What is the probability of rolling six dice and
   (a) getting a sum of 6?
   (b) getting a sum of a 7?
   (c) getting a sum of 10?
   (d) having all six numbers be equal?

(e) having all six numbers be different?

5. *Non-transitive dice.* Cosider the following four unusual dice.



You pick one die, and then I will pick another. We will each roll our chosen die and the larger number wins. Do you want to play?

Use the 6×6 tables to figure out who wins, when you pair, for example, die A and die B.

Answer: Each die is beaten with probability 2/3 by the die to its left. I.e. B is beaten by A, C by B, D by C, and A by D. This problem can be pushed further with the follow-up question: Three dice A, B, and C have the property that A beats B, B beats C, and C beats A all with the same probability. Can you find them?

A vs B	3	3	3	3	3	3
0	B					
0	B					
4	A					
4						
4						
4						

A vs C						

A vs D						

B vs C						

B vs D						

C vs D						

6. Two cards are dealt off the top of a well-shuffled deck.
   (a) Find the chance that the second card is an ace.
   (b) Find the chance that the second card is an ace, given that the first card is an ace.
   (c) Find the chance that the second card is an ace, given that the first card is a king.
   (d) Find the chance that the first card is a king and the second card is an ace.
7. Three cards are dealt from a well-shuffled deck.
   (a) Find the chance that all of the cards are diamonds.
   (b) Find the chance that none of the cards are diamonds.
   (c) Find the chance that the cards are not all diamonds.
8. *Random Trisection.* A deck of cards is randomly cut into three piles. I bet that at least one of the cards on the top of a pile is a "face card" i.e. a Jack, Queen, or King. Do you want to take this bet?

   Answer: The probability is $1 - 40/52 \cdot 39/51 \cdot 38/50 = 47/85 \approx 0.55$. *Don't take the bet.*
9. *Heads I Win.* Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that
   (a) the game never ends?
   (b) the first player wins?
   (c) the second player wins?

   Answer: The probability that the first player wins can be written as an infinite sum $1/2 + (1/2)^3 + (1/2)^5 + \cdots$ *which sums to* $2/3$. *Aternatively, if* p *is the probability that the next player to move gets a head before the other player does, then* $p = 1/2 + (1/2)^2 \cdot p$, *which can be solved to get* $p = 2/3$.
10. *Three-way duel.* Alexander Hamilton, Aaron Burr, and Thomas Jefferson fight a 3-cornered pistol duel. All three know that Alexander Hamilton's chance of hitting any target is 1/3, while Aaron Burr *never* misses, and Thomas Jefferson has a 0.5 chance of hitting any target. The way the duel works is that each person is to fire at their choice of target, starting with Alexander Hamilton, and proceding to Aaron Burr, then Thomas Jefferson, then Alexander Hamilton again, etc. (unless someone is hit, in which case they don't shoot), continuing until one person is left unhit. What is Alexander Hamilton's strategy?

    Experiment with different strategies, and simulate the shooting using dice. You can simulate an event with probability 1/3 by tossing one die and seeing if the number shown is 1 or 2, say. Likewise, you can come up with ways to simulate a 1/2 probability event (you don't need a coin, you can still use a die).

    Here are some possible strategies for Alexander Hamilton: shoot at Aaron Burr first; shoot at Thomas Jefferson first; try something else. Choose a strategy, and try simulating the duel. See if you can experimentally estimate the probability that Alexander Hamilton survives, using various strategies.

Try to justify your experimental conclusions with calculations.

Answer: The best strategy for Alexander Hamilton is to shoot in the air. Which is what he did in the real duel, which was actually just a 2-way duel against Aaron Burr. Intuitively, we can see that if he shoots at Thomas Jefferson, it is better not to hit him, since if he hits Jefferson, he will surely die. So shooting in the air is better than shooting Jefferson. Shooting in the air is also better than shooting Aaron Burr, but I don't have a simple way to see this. It is possible to calculate the probabilities using the same methods as in the warm-up problem "Heads I Win". I calculated that if Hamilton shoots at Jefferson, he will live with probability 2/9, and if he shoots at Burr, he will live with probability 11/36, and if he shoots in the air, he will win with probability 1/3, but please be skeptical of my answers!

References: All these problems are from Paul Zeitz, Math Professor, University of San Francisco. The non-transitive dice problem is in *Solve This!* by James Tanton.