Modular Arithmetic

November 11, 2017

Warm-up

Recall: For two numbers A and B, we say that $A \equiv B \mod 5$ if A and B have the same remainder when divided by 5.

- $8 \equiv 23 \mod 5$ because the remainder of 8 divided by 5 is 3, and the reminder of 23 divided by 5 is also 3.
- $8 \not\equiv 14 \mod 5$ because the remainder of 8 divided by 5 is 3, but the remainder of 14 divided by 5 is 4, NOT 3.

1. Compute these sums. Hint: you don't need to do a lot of arithmetic.
   (a) $423 + 577 \mod 10$
   (b) $56 + 89 \mod 10$
   (c) $892 + 9828 \mod 5$
   (d) $923 + 725 \mod 3$

2. Compute these products. Hint: be lazy.
   (a) $4893 \times 49042 \mod 10$
   (b) $3982734 \times 2398739 \mod 10$
   (c) $78 \times 23 \mod 5$
   (d) $3874 \times 3284 \mod 3$

3. This is a magic trick performed by two magicians, A and B, with one regular, shuffled deck of 52 cards. A asks a member of the audience to randomly select 5 cards out of a deck. The audience member who we will refer to as C from here on then hands the 5 cards back to magician A. After looking at the 5 cards, A picks one of the 5 cards and gives it back to C. A then arranges the other four cards in some way, and gives those 4 cards face down, in a neat pile, to B. B looks at these 4 cards and then determines what card is in Cs hand (the missing 5th card). How is this trick done?

   See https://math152.wordpress.com/2008/10/30/modular-arithmetic-and-a-cool-card-trick/ for an explanation.

1 Last digits

1. (a) What is the last digit of $14,306 + 908,797$? Can you find the answer quickly, without doing the whole addition problem?
   (b) What is the last digit of 5589×4523?
   (c) What is the last digit of $413 \times 5967 \times 4534$?
2. What is the last digit of 9^{99}? Remember, 9^{99} means we multiply 9 by itself 99 times. Hint: try to find a pattern by finding the last digit of 9^1, 9^2, 9^3, etc.
3. What is the last digit of 3^{2017}?
4. What is the last digit of 2^{100}?
5. Find the last digit of the number $1^2 + 2^2 + 3^2 + 4^2 + \cdots 99^2$.

Extra Problems:

6. Find the remainder of 2^{100} when divided by 3.
7. Find the remainder when the number 3^{2017} is divided by 7.
8. Find the remainder when the number 9^{100} is divided by 8.
9. Find the remainder when the number $1989 \times 1990 \times 1991 + 1992^3$ is divided by 7

2 Square Numbers

1. Fill in the table to find the values of the square numbers mod 4.

Number X	Square Number X^2	$X^2 \mod 4$
1	1	
2	4	
3	9	
4		
5		
6		
7		
8		
9		
10		

2. Do you notice a pattern in the table above? What can you say about the square of an odd number mod 4? The square of an even number?
3. Is the number 114502909924083 a perfect square? Why or why not?
4. Is it possible to find two numbers, whose squares add up to 74? (The square of a number means the number times itself. For example, the square of 9 is 81.)
5. Is it possible to find two numbers, whose squares add up to 1111? Hint: what is $1111 \mod 4$? What are the square numbers mod 4?
6. Can the sum of two perfect square numbers be a perfect square number? A perfect square number is a number like 4 or 9 that is the square of another number.
7. Can the sum of squares of two odd numbers be a perfect square?
8. Can the sum of squares of three odd numbers be a perfect square?
9. Can the sum of squares of five consecutive numbers be a perfect square?

Extra Problems

10. Start with 5 pieces of paper. At each step, choose one piece of paper and cut it into 4 pieces. Prove that you will never be able to get exactly 100 pieces of paper this way.

11. Is it possible to find a number x so that the following facts are true?

    (a) $x \equiv 5 \mod 6$

    (b) $x \equiv 3 \mod 10$

12. Is it possible to find a number x so that the following facts are true?

    (a) $x \equiv 7 \mod 9$

    (b) $x \equiv 5 \mod 12$