CHMC: Lights Out

9/16/17

1 Introduction

Lights Out is a game played on a square grid of lights. Some of the lights are on, some of the lights are off, and the goal is to turn all of the lights off by pressing a series of buttons.

It turns out that we can analyze this game in a very satisfying mathematical framework involving arithmetic mod 2, systems of equations, and matrices.

The second section will introduce the puzzle and the rules. The third section introduces a technique we'll use quite a bit called *Gaussian elimination*, that allows us to solve general systems of equations. Our analysis of Lights Out will eventually come down to solving systems of equations.

The fourth section introduces an arithmetic framework that will let us transform this puzzle into a system of equations, and the last section explores these systems.

The fifth section brings together all of the tools we've developed and applies them to solving the Lights Out puzzle. It turns out that the techniques are very tedious to do by hand for large puzzles, but 3×3 boards aren't bad at all.

The sixth and final section explores a generalization of this puzzle to a torus. Luckily our techniques still apply, but the solutions can change.

2 The Puzzle

As mentioned, the puzzle consists of a square grid of lights. The starting grid may have some lights on, and some lights off, and the goal is to turn all of the lights off.

The allowed moves are pressing the lights themselves: if a light is on, pressing it will turn it off; if a light is off, pressing it will turn it back on. The catch is that pressing a light will turn on (or off) the lights directly above and below it, as well as to the left and right of it.

As an example, consider this sample board (where O denotes a light that's on):

O		O
	O	O

Pressing the center light will result in the board

O	O	O
O		
	O	

.

Exercise 2.1 Solve the boards:

O	O	O
O		
	O	

and

		O

.

The 3×3 case is relatively simple; the original puzzle involved a 5×5 grid of lights.

Exercise 2.2 Solve the board

O	O	O		
		O		
		O	O	O

.

Having seen (and hopefully solved) a few boards, it's not too clear that a random board is solvable. In fact, every $2 \times 2, 3 \times 3, 6 \times 6, 7 \times 7$, and 8×8 board is solvable (together with some larger grids), but there are some 4×4 and 5×5 boards that are not solvable.

Exercise 2.3 Show directly that every 2×2 board is solvable. In other words, show that all of the $2^4 = 16$ 2×2 grids are solvable (you don't need to actually check all 16, since many of these boards will appear in solving others).

We could do a similar exercise to show, by hand, that all 3×3 boards are solvable, but we won't do that now. In a few sections, though, we'll have a nice technique for solving any board (that doesn't come down to trying different button combinations).

3 Gaussian Elimination

Consider the system of equations

$$\begin{aligned} x + 3y &= -2, \\ 2x - 5y &= 1. \end{aligned}$$

A standard method would be to solve the first equation for x, then plug it into the second equation: we have $x = -2 - 3y$, and plugging this back into the second equation gives

$$\begin{aligned} 2(-2 - 3y) - 5y &= 1 \\ -4 - 6y - 5y &= 1 \\ -11y &= 5 \\ y &= \frac{-5}{11}. \end{aligned}$$

Plugging this value of y back into the first equation gives us our solution for x:

$$\begin{aligned} x + 3\left(\frac{-5}{11}\right) &= -2 \\ x &= -2 + \frac{15}{11} \\ x &= \frac{-7}{11}. \end{aligned}$$

Exercise 3.1 Solve the system

$$\begin{aligned} 3x + 7y &= 1 \\ -2x + 3y &= 3. \end{aligned}$$

While this method is valid, it becomes unfeasible for large systems. For example, try solving

$$\begin{aligned}
x_1 + x_3 + x_4 + x_6 + x_8 + x_9 &= 1\\
x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 &= 0\\
x_2 + x_3 + x_4 + x_5 + x_6 + x_8 + x_9 &= 0\\
x_1 + x_2 + x_3 + x_7 &= 1\\
x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 + x_9 &= 1\\
x_1 + x_2 + x_3 + x_5 + x_6 + x_8 + x_9 &= 0\\
x_1 + x_2 + x_4 + x_5 + x_6 + x_8 + x_9 &= 1\\
x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 + x_9 &= 0\\
x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9 &= 0.
\end{aligned}$$

Systems like this will actually appear later, so we'll need a much more efficient way of solving these. This is where Gaussian elimination comes into play.

Let's take another look at the system

$$\begin{aligned}
x + 3y &= -2,\\
2x - 5y &= 1.
\end{aligned}$$

Let's multiply the first equation by 2:

$$\begin{cases} 2x + 6y &= -4,\\ 2x - 5y &= 1, \end{cases}$$

and subtract from the second equation the first equation, giving

$$\begin{aligned}
&\begin{cases} 2x + 6y &= -4,\\ 2x - 2x - 5y - 6y &= 1 - (-4), \end{cases}\\
\implies &\begin{cases} 2x + 6y &= -4,\\ -11y &= 5, \end{cases}
\end{aligned}$$

We can now solve for y in the second equation,

$$\begin{cases} 2x + 6y &= -4,\\ y &= \frac{-5}{11}, \end{cases}$$

, and subtracting from the first equation 6 times the second equation, we're left with

$$\begin{cases} 2x &= -4 - \frac{-30}{11}, \\ y &= \frac{-5}{11}, \end{cases}$$
$$\implies \begin{cases} 2x &= \frac{-14}{11}, \\ y &= \frac{-5}{11}, \end{cases}$$
$$\implies \begin{cases} x &= \frac{-7}{11}, \\ y &= \frac{-5}{11}, \end{cases}.$$

We get the same result, but thinking of the process in terms of row operations lets us solve these systems symbolically. The technique is to write everything as an *augmented matrix*, and add multiples of rows with each other. To get the augmented matrix of a system, we align the variables in different equations so that all of, say, the xs are above or below each other, all the ys are above or below each other, etc. and write down their coefficients (including 0) in a square table. We next draw a vertical line to the right of this table, and then write down the constants to the right of the equalities.

For example, the system

$$\begin{aligned} x + 3y &= -2, \\ 2x - 5y &= 1, \end{aligned}$$

is written as

$$\left|\begin{array}{cc|c} 1 & 3 & -2 \\ 2 & -5 & 1 \end{array}\right|.$$

To modify an augmented matrix, we can multiply a whole row by a non-zero number, and we can add rows to each other. The goal is to end up with a square of 1s and 0s on the left, from which we know that the variable corresponding to a 1 in the left-hand side equals the number in the same row on the other side of the vertical line. Lets see this in action.

The first operation we did to the system was multiply the first row by 2, resulting in

$$\left|\begin{array}{cc|c} 2 & 6 & -4 \\ 2 & -5 & 1 \end{array}\right|,$$

and subtracting the first row from the second row gives

$$\left|\begin{array}{cc|c} 2 & 6 & \text{-4} \\ 0 & \text{-11} & 5 \end{array}\right|.$$

We want a 1 in the bottom middle entry, so dividing by -11 gives

$$\left|\begin{array}{cc|c} 2 & 6 & \text{-4} \\ 0 & 1 & \frac{-5}{11} \end{array}\right|.$$

Finally, subtracting 6 times the second row from the first row gives

$$\left|\begin{array}{cc|c} 2 & 0 & \frac{-14}{11} \\ 0 & 1 & \frac{-5}{11} \end{array}\right|.$$

Since we want a 1 in the top left corner, we divide the first row by 2:

$$\left|\begin{array}{cc|c} 1 & 0 & \frac{-7}{11} \\ 0 & 1 & \frac{-5}{11} \end{array}\right|.$$

Notice that this agrees with our original solution, and is computationally easier to manage.

Exercise 3.2 Solve the system

$$\begin{aligned} 3x + 2y + z &= 1 \\ y - 3z &= -2 \\ x + 5z &= -1, \end{aligned}$$

using Gaussian elimination. The process is the same as for a system of two equations, except now we'll be working with a 3×4 augmented matrix.

It's even possible for a system to not have any solutions to a system of equations!

Exercise 3.3 Consider the system of equations

$$\begin{aligned} 2x - 3y &= 15, \\ 2x - 3y &= 14. \end{aligned}$$

Just by looking at the equations, why is it impossible to have a solution, say, $x =$something and $y =$something? What happens if you try to apply Gaussian elimination to this system?

4 All about $\mathbb{Z}_2$

We can think of turning a light on or off as a type of arithmetic. We'll write $\mathbb{Z}_2 = \{0, 1\}$ with the following addition and multiplication rules:

$$\begin{aligned} 0 + 0 &= 0, \\ 0 + 1 &= 1, \\ 1 + 0 &= 1, \\ 1 + 1 &= 0; \\ 0 \cdot 0 &= 0, \\ 0 \cdot 1 &= 0, \\ 1 \cdot 0 &= 0, \\ 1 \cdot 1 &= 1. \end{aligned}$$

Exercise 4.1 In $\mathbb{Z}_2$, what does the number 2 look like? What about 3? How can you generalize this? Hint: write these numbers as a sum of ones.

Exercise 4.2 What does -1 look like in $\mathbb{Z}_2$?

Exercise 4.3 In $\mathbb{Z}_2$, what is $2 \cdot 3$? What about $3 \cdot 7$? How can you generalize this?

Exercise 4.4 How do lights turning on and off relate to arithmetic in $\mathbb{Z}_2$?

In the last section we looked at systems of equations, where the coefficients of the variables were integers, and the solutions could be rational numbers. What happens if we only allow coefficients from $\mathbb{Z}_2$, and we only look for solutions that are also in $\mathbb{Z}_2$? The same technique works!

Exercise 4.5 Write the system

$$\begin{aligned} w + x + z &= 0 \\ w + y &= 1 \\ x + y &= 0 \\ w + z &= 0, \end{aligned}$$

as an augmented matrix. Apply Gaussian elimination to this matrix; are there solutions for w, x, y, z? If so, what are they? Remember that everything is being done in $\mathbb{Z}_2$.

5 Solutions to Lights Out

We now have all of the tools to "mathematize" Lights Out.

To begin, note that a Lights Out board can be interpreted as a matrix of 0s and 1s. For example, the board

O		O
	O	O

is equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lets call the board B, and the corresponding matrix $[B]$.

One key observation is that pressing the middle button on B is equivalent to adding the matrices

$$[B] + [1,1] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$[i, j]$ will be our notation for matrix consisting of a "cross of 1s" centered at the ith row and jth column.

Exercise 5.1 Justify this last statement. Does the resulting matrix correspond to what happens on the actual Lights Out board?

Exercise 5.2 Suppose you pressed the top right button of B instead of the middle button. What matrix would you have to add to $[B]$ to get the resulting Lights Out board?

It will help to write $[B]$ as a vector, where we stack the rows on top of each other. In this case, we'll have

$$[B] = (1,0,1,0,1,1,0,0,0)^t = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

To save space, we'll use the first vector for $3^2, 4^2$, and higher dimensional vectors. Since 2^2 terms aren't too many, we'll usually write the vertical representation for these vectors.

Let's examine the 2×2 case. As matrices, pressing the button in the top left corner corresponds to the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

We can encode pressing this button as multiplying the matrix by 0 or 1, and in general we'll write $b_{i,j} = 0, 1$ depending on whether or not we press the button in the ith row and jth column.

The matrices corresponding to button presses in the 2×2 case are

$$[1,1] = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, [1,2] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, [2,1] = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, [2,2] = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Exercise 5.3 Check this for yourself. Start by writing out the square matrix you would have to add to a board $[B]$, and then stack the rows.

Before we can turn solving this puzzle into a system of equations, we'll need one more important observation: solving a board is the same as turning the solved board into the original board. In other words, if we have a sequence of button presses that will solve a board, pressing the same buttons will turn an empty board into the original configuration B.

Exercise 5.4 Does the order you press the buttons in matter? Why or why not?

As an example, suppose we want to solve the board

	O
O	

.

We start by writing this as a matrix

$$[B] = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

We're looking for a sequence of button presses $b_{i,j}$ that result in $[B]$, i.e. we want $b_{i,j}$ in $\mathbb{Z}_2$, for $i, j = 1, 2$ such that

$$b_{1,1}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + b_{1,2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + b_{2,1}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + b_{2,2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

which we can rewrite as the system

$$\begin{aligned} b_{1,1} + b_{1,2} + b_{2,1} &= 0 \\ b_{1,1} + b_{1,2} + b_{2,2} &= 1 \\ b_{1,1} + b_{2,1} + b_{2,2} &= 1 \\ b_{1,2} + b_{2,1} + b_{2,2} &= 0. \end{aligned}$$

Here, the $b_{i,j}$ are variables that we want to solve for.

Exercise 5.5 Write this system as an augmented matrix. Are there solutions for the $b_{i,j}$? If you press the corresponding buttons on the original board

	O
O	

,

do you actually have a solution to the puzzle?

Exercise 5.6 For a 3×3 Lights Out puzzle, how many matrices $[i, j]$ will there be? What about an $n \times n$ board?

Exercise 5.7 Write out the matrices $[i, j]$ for a 3×3 Lights Out puzzle. It may help to write the square matrices first, and then stack the rows.

Exercise 5.8 What is a/the solution for the Lights Out puzzle

O		O
O	O	
O		

?

Why?

It turns out that *every* 3×3 Lights Out puzzle has a solution, but not every 4×4 or 5×5 puzzle is solvable. Unfortunately we won't be able to prove that in this worksheet, but similar techniques to what we've used here are employed. Part of the issue is that, from a previous exercise, we know that for a $n \times n$ board, we'll need n^2 matrices $[i, j]$. This means we'll need to solve a system of n^2 equations in n^2 unknowns; even for $n = 4$ this is tedious (and not worth it) to do by hand.

We can explore another extension of this problem though!

6 Lights Out on a torus

Suppose now that the game is played on a torus. This means that, when you press one of the top buttons, say, some of the bottom lights will also be affected. For example, normally if we press the top left button on a 3×3 board we would end up with the board

$\longrightarrow$

O	O	
O		

,

whereas on a torus we would get

$\longrightarrow$

O	O	O
O		
O		

.

Exercise 6.1 The matrices $[i, j]$ for a 2×2 board are the same if the puzzle is played on a torus, but different for a 3×3 board. What are the new matrices $[i, j]$?

In practice though, we end up with a system of equations that can, again, be reduced using Gaussian elimination. Even though the situation has changed and the matrices look different, the method is exactly the same.

Exercise 6.2 Is the board

O	O	O
O		
O		

solvable on a torus? Why or why not?

Exercise 6.3 Come up with your own torus Lights Out puzzles and try to solve them. Can you find puzzles that don't admit solutions?

References

[1] Website: http://mathworld.wolfram.com/LightsOutPuzzle.html

[2] Anderson, Feil; "Turning Lights out with Linear Algebra"