Farey Sequences and Ford Circles

Based on notes from Dana Paquin and from Joshua Zucker and the Julia Robinson Math Festival.

1 Organizing Fractions

On the first row of the sequence of Farey fractions, we have two fractions: 0 and 1, which we write in lowest terms as $\frac{0}{1}$ and $\frac{1}{1}$.

On each row after that, we insert all the fractions between 0 and 1 with a denominator one higher than the biggest denominator of the previous line. That is, line 2 gets the fractions with denominator 2, line 3 gets the fractions with denominator 3, and so on.

Here are the first few rows.

$$\frac{0}{1}, \quad \frac{1}{1}$$

$$\frac{0}{1}, \quad \frac{1}{2}, \quad \frac{1}{1}$$

$$\frac{0}{1}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{1}{1}$$

$$\frac{0}{1}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{1}{1}$$

Write out the next few rows.

2 Fraction Patterns

1. Magical Property #1: Look at pairs of adjacent fractions. What do you notice about them? (Hint: Cross-multiply)

2. Magical Property #2: When you insert a new fraction, look at the fractions to its right and left. How does the numerator and denominator of the new fraction relate to its old neighbors?

3 Why do these patterns hold?

3. Magical Property #1 can be proved from Pick's Theorem, which relates the area of a lattice polygon to the number of interior lattice points and the number of boundary lattice points.

   (a) Write down Pick's Theorem from last time or ask someone what it says.

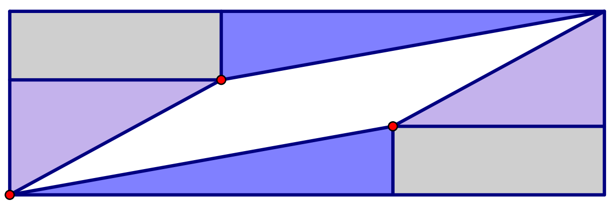
   (b) Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are two successive terms of Fn. Let T be the triangle with vertices $(0,0)$, (a,b), and (c,d).

   (c) Show that T has no lattice points in its interior, i.e. $I(T) = 0$.

   (d) Show that the only boundary points of T are the vertices of the triangle, i.e. $B(T) = 3$.

   (e) Conclude, using Picks Theorem, that $A(t) = \frac{1}{2}$, where $A(T)$ means the area of T.

   (f) Use geometry to show that $A(T) = \frac{1}{2}(bc - ad)$. Hint: use this picture.



   (g) Use the above facts to show that Magical Property #1 holds.

4. Suppose that $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$ are three consecutive Farey fractions. Magical Property #2 can be proved from Magical Property #1, as follows.

   (a) Write down the equations for Magical Property #1 for $\frac{a}{b} < \frac{c}{d}$ and for $\frac{c}{d} < \frac{e}{f}$, and set these equations equal to each other.

   (b) Rearrange the resulting equation to prove Magical Property # 2.

5. Can you show that all the fractions that have the relationship of Magic Property #1 eventually show up as neighbors in a Farey sequence? If not, can you find an example that doesn't work? Hint: show that if $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, then $d \geq b + f$.

4 Ford Circles

6. Place the fractions along a number line. You may just want to write just the first few rows of the Farey fractions. On top of each fraction, like a wheel resting on the number line, draw a circle. If the fraction is $\frac{a}{b}$ then the circle should have radius $\frac{1}{2b^2}$. If you draw your diagram roughly to scale, you may make an amazing discovery! These circles are called the Ford circles.

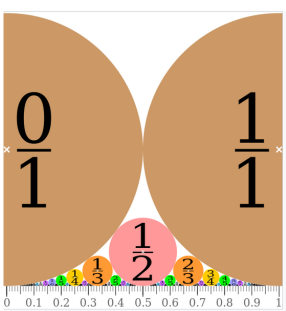
7. First let's investigate tangent circles in general. Draw a horizontal line, and two circles that touch the line and touch each other, one of radius r and one of radius R. Determine the distance between the two points where the circles touch the horizontal line in terms of r and R.

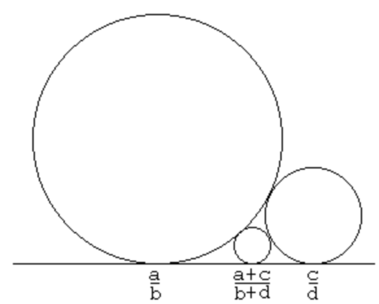
8. Suppose that you have a circle of radius $\frac{1}{2b^2}$ resting above $\frac{a}{b}$ and a circle of radius $\frac{1}{2d^2}$ resting above $\frac{c}{d}$ and suppose that these two circles happen to be tangent. Use the previous step to show that $\frac{a}{b}$ and $\frac{c}{d}$ have Magic Property #1.

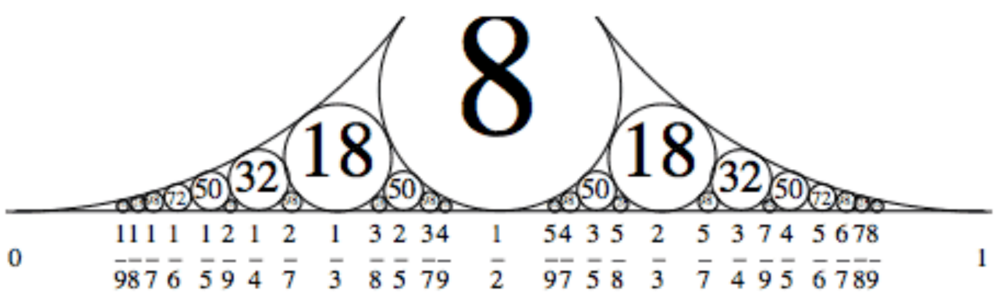
9. Reverse your steps to show that if $\frac{a}{b}$ and $\frac{c}{d}$ have Magic Property #1, then the circles of radius $\frac{1}{2b^2}$ and $\frac{1}{2d^2}$, respectively, above them, are tangent.

10. Now what can you prove about the Ford circle for a newly inserted fraction in relation to the Ford circles of its neighbors?

Here are some hints in the way of pictures:







Figures from http://blog.wolfram.com/2010/06/16/the-circles-of-descartes/, https://en.wikipedia.org/wiki/Ford_circle, and http://www.cut-the-knot.org/proofs/fords.shtml.