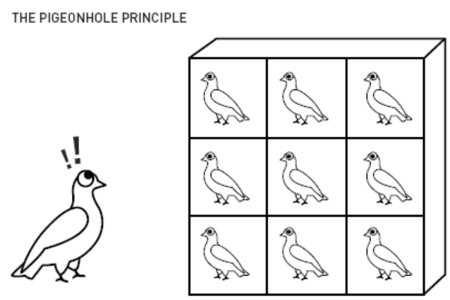
Pigeonhole Principle

November 19, 2016

1 Warm-up problems

1. I own 7 pairs of socks and each pair is a different color. If all 14 socks are loose in the dryer, how many will I have to pull out to guarantee that I get at least two of the same color?
2. (a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? Suit means hearts or diamonds or clubs or spades.
   (b) How many cards must be selected to guarantee that at least three hearts are selected?
3. Show that in any group of five people, there are two who have an identical number of friends within the group. (Friendship is mutual – if A is B's friend, then B is A's friend.)

> **Pigeonhole Principle:**
>
> (a) If you put $n + 1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than one pigeon.
>
> (b) If you put $kn + 1$ or more pigeons into n pigeon holes, at least one pigeon hole must contain more than k pigeons.



2 More Pigeonhole Problems

4. Of 40 children seated at a round table, more than half are girls. Prove that there are two girls who are seated diametrically opposite each other.
5. Eight chairs are set around a circular table. On the table are name placards for eight guests. After the guests are seated, it is discovered that none of them are in front of their own names. Show that the table can be rotated so that at least two guests are simultaneously correctly seated.
6. Several soccer teams enter a tournament in which each team plays every other team exactly once. Show that, at any moment during the tournament, there will be two teams which have played, up to that moment, the same number of games.

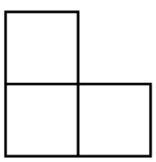
7. Given 12 integers, show that two of them can be chosen whose difference is divisible by 11.
8. Suppose we are given 10 different numbers from 1, 2, 3, . . . 99. Prove that there exists two disjoint subsets with the same sum. For example, if we are given the ten numbers $\{1, 2, 7, 11, 23, 31, 35, 48, 83, 91\}$ then

$$2 + 11 + 23 = 35 + 1$$

[Hint: How many subsets of ten numbers are there? How many different sums are there?]

3 Hard Problems

9. (a) Given five points in a square of side length 1, show that two of the points must be no more than $\sqrt{2}/2$ apart.
   (b) Given five points in an equilateral triangle of side length 2, show that two of the points must be no further than 1 apart.
   (c) Given nine points in a square of side length 2, show that three of the points must form a triangle whose area is not more that 1/2.
10. What is the largest number of squares on an 8×8 checkerboard which can be colored green, so that in any arrangement of three squares (a ”tromino” as drawn below), at least one square is not colored green? (The tromino may be appear as in the figure or it may be rotated through some multiple of 90 degrees.)



11. Given eight different positive integers, none greater than 15, show that at least three pairs of them have the same positive difference. (The pairs may overlap that is, two pairs or all three pairs may contain the same integer.)
12. Prove that among any six people, there are either three people who all know each other or three people who are all strangers to each other. (Assume that if person A knows person B, then person B also knows person A.)
13. Five lattice points are chosen on an infinite square lattice. Prove that the midpoint of one of the segments joining two of these points is also a lattice point.
14. Prove that there exist two powers of 2 whose difference is a multiple of 2016.
15. Prove that you can choose a subset of ten given integers such that their sum is divisible by 10.
16. Given 11 different positive integers, none greater than 20, prove that two of these can be chosen, one of which divides the other.

These problems are from *Mathematics Circles: the Russian Experience* by Fromkin, Genkin, and Itenberg and from *A Decade of the Berkeley Math Circle* by Stankova and Rike, and from UNC's Problem Solving Seminar.