Whether it's sharing out pizza or shopping with their parents, children encounter mathematics every day. But how can we bridge the gap between these real-world experiences with maths and the abstract concepts taught as part of the curriculum? **Ana Helvia Quintero** and **Héctor Rosario** explain.

Making maths make sense

Experience teaches us that a great number of students do not learn many of the mathematical concepts introduced in primary school. Nevertheless, these same children use mathematics every day. Without knowing much about the formal structure of rational numbers, for example, you can bet most children know how to fairly divide a pizza or a chocolate bar!

This realisation led us to develop a new, constructivist approach to our maths teaching, based on Freudenthal's philosophy of Realistic Mathematics Education.[1] In this approach, real-life situations play a key

role in bridging the gap between the informal mathematics that children learn from their day-to-day experiences and bring to the classroom, and the formal mathematical world taught in schools.

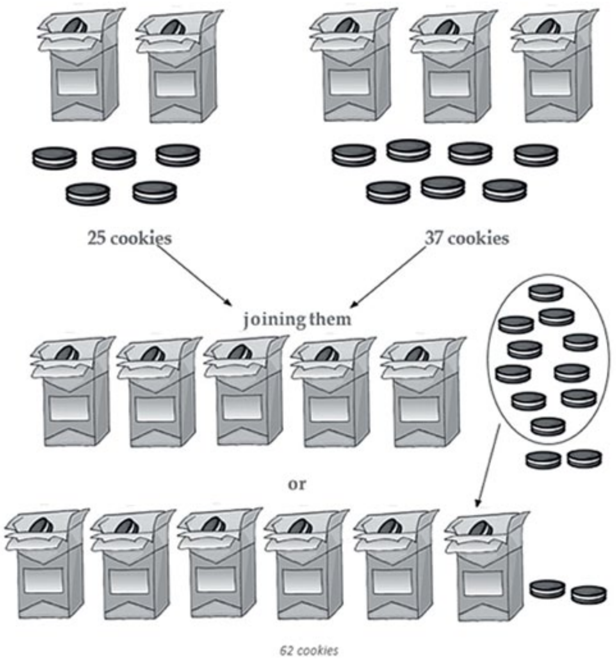
How real-world maths works

The approach works mainly through the use of models, or representations of the problem posed. Students can play an active role in developing models, so they do tend to evolve throughout the teaching and learning process.

There are many models that teachers can include in their teaching practice to help students understand different mathematical concepts. For instance, children's knowledge of how products are packaged can be used to introduce the following situation as a model for our number system and as a way to understand the addition and subtraction algorithms.

A shop sells cookies individually and in boxes of ten cookies. A girl goes into the shop – she asks first for 25 cookies and then for 37 more cookies. If her requests are joined in one order, how many bags of ten cookies and how many loose cookies will she get?

The following picture illustrates a mental construction that a student might make or that a teacher might facilitate:



From this model, students can begin to understand the process of carrying in the typical addition algorithm by relating it to the real-world concept of grouping.

What should maths teaching look like?

The teaching of mathematics usually follows the structure of mathematics; we call this the 'logic of mathematics'. However, the logic of mathematics is not the same as the logic of *learning* mathematics. Some concepts that precede others in maths succeed them in the learning of mathematics. For instance, the concept of a segment is used to define a triangle in Euclidean geometry, yet students understand more easily the idea of a triangle than that of a segment. Hence, it is pedagogically sounder to teach the concept of a triangle before that of a segment.

Yet it is often the case that the teaching of mathematics follows the structure of the discipline and not the structure of how it can be learned most effectively. Our approach aims to follow the logic of learning mathematics instead of the logic of mathematics. How? Let us consider the case of addition of heterogeneous fractions – that is, fractions with different denominators.

Since equivalent fractions are needed to add heterogeneous fractions, the structure of the discipline dictates that:

Equivalent fractions → Addition of heterogeneous fractions

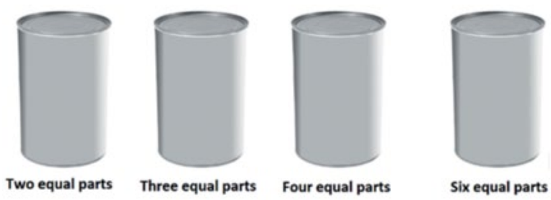
Notice that equivalent fractions have no context in this approach. They are needed as a prerequisite to being able to add heterogeneous fractions. Yet, it is often the case that students better understand addition of fractions than the concept of equivalent fractions. Indeed, the need of using equivalent fractions arises when we add heterogeneous fractions.

Hence, the logic of learning mathematics becomes:

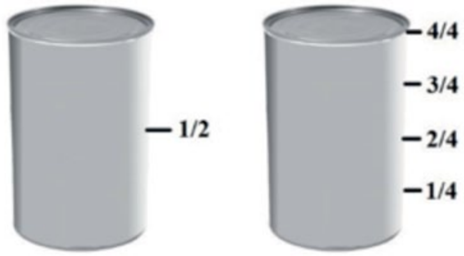
Intuitive addition of heterogeneous fractions → Necessity of equivalent fractions → Equivalent fractions → Addition of heterogeneous fractions

Let's illustrate this through a lesson in which we teach this according to the cognitive development of the concepts.

Jorge, Mary and Laura are camping. Laura loves cooking and she brought her cooking book, but she forgot her measuring cups. Mary came up with the idea of using some empty cans as measuring cups by marking different fractions. Here are the cans and the fractions that they want to mark on each can. Help them mark the cans.



A recipe asks for 1/2 cup of orange juice and 3/4 cups of water. Jorge says they can pour both liquids into the same can. Mary says it *cannot be done because it would spill. Who is correct? Why?*



Mary convinces Jorge by drawing the following sketch:

He notices that 1/2 is more than 1/4. So when the can is 3/4 full, there is only 1/4 of a cup empty, and pouring 1/2 of a cup would spill the orange juice.

Laura wants to accommodate the two liquids in two cans, but she wants to fill up one can. How much liquid would be left for the second can?

(There are several ways to solve this problem. In most cases, it will be evident that 1/2 = 2/4, leading to the idea of equivalent fractions. However, there are other ways to solve this problem without the need for formally finding equivalent fractions.)

While camping, they find some coconut trees. Jorge cuts down two coconuts. He extracts their water and pours the water from each coconut into different cans. One of the coconuts fills 1/2 of the can, and the other one fills 1/3 of another can. Is there enough room in one can to pour the water from both coconuts? Why? How much coconut water would there be in both cans if they were combined?

Analysing these two situations, we observe that to add heterogeneous fractions, it is necessary to express them with a common denominator, which requires the concept of equivalent fractions.

This example shows that we must keep in mind the cognitive development of concepts during curriculum development. By presenting similar situations to pupils, we promote the construction of concepts that do not necessarily follow the logic of the discipline.

Admittedly, more research is needed to discover the cognitive development of mathematical concepts. It is also necessary to research how teaching can be transformed to promote learning following this development.

Making the most of technology

As we have shown, models are an invaluable tool for linking the abstract mathematical world

to the real-world maths pupils find so intuitive. The power of technology gives students access to a wide range of visual models. Simulations, for example, allow us to vary different factors in a real-world situation and see the effects this generates. Pupils can see the role variables play in the situation and begin to understand the relationship between them, giving meaning to otherwise abstract formulas. Spreadsheets are another tool which is particularly useful for exploring variables, as these give a quick visual overview of what happens as variables change.

Technology also allows us to perform arithmetic, algebraic and statistical calculations with great ease. This frees students from focusing on the mechanical aspect of analysis so they can focus their attention on what is essential for problem solving: interpretation, comprehension and analysis.

An area of particular interest to teachers looking to combine technology and mathematics is coding—now a requirement of the new primary computing curriculum. In coding, not only must students learn how an algorithm works, but they need to learn how to organise the steps of the algorithm in order for the computer to understand their commands. This requires developing good reasoning and problem-solving skills.

Of course, no teacher would agree that calculators replace the learning of algorithms and other mathematical concepts that allow students to perform mental arithmetic and estimations. It is the same with other forms of technology. First, pupils must learn the basic concepts without technology—like learning what an average, an integral or a matrix is. They should also learn how to perform basic operations. Once they are able to do this, technology can help in situations requiring time-consuming or tedious calculations, some of which might be impossible to do by hand.

Aliens, dragons and princesses can make real-world maths too!

When selecting problems for children to solve, we should remember that the 'realistic' concepts students bring to the classroom involve aliens flying across galaxies; wizards, gnomes and elves inhabiting enchanted forests; dragon-fighting knights rescuing princesses from mystical castles; superheroes saving the world; and much, much more! We can use these elements and concepts to create problems and puzzles that capture students' attention and feed their imagination.

Through decades of observation, we have witnessed children exhilarated by the excitement that comes from solving mathematical and logic puzzles, many of them imbued with the spirit of the aforementioned fantastical concepts. The idea, however, is not to 'make maths fun', but to simply reveal what mathematics is: an intellectually stimulating and gratifying adventure!

Puzzles come in all sorts of forms and

contexts, and can fit into the curriculum of any classroom at any level. It is, of course, always a good idea to choose puzzles where students can apply similar techniques to those they have used to solve another problem. Puzzles can reveal children's hidden mathematical talents and can also stimulate much deeper thinking about the nature and purpose of maths.

For example, consider this liars and truth-tellers puzzle based on the Knights of the Round Table:

Three knights come to King Arthur to inform him about the slaying of a dragon.

'Sir Lancelot killed the dragon,' said Sir Gawain.

'Sir Tristan killed the dragon,' said Sir Lancelot.

'I killed the dragon,' said Sir Tristan.

If only one knight told the truth, then who killed the dragon?

Only Gawain could have told the truth and hence, Lancelot must have killed the dragon. (Why a knight would lie is an interesting moral question!).

Such puzzles are an excellent introduction to logical reasoning, and young students seem to revel in them. Even those who come into the classroom with resistance to maths feel safe attacking these sorts of problems, perhaps because they do not see the connection to mathematical thinking. Take advantage of this situation and guide them into exploring the connections between logical and mathematical thinking with questions like: Is this maths? If it is not, what is it? If it is, how is it so? You may need to delve more deeply into the subject and ask: What is mathematics? Is it about numbers? Is it about patterns? These are open questions worth pondering from an early age.

Creating new assessments

A new learning vision requires new assessments.[2] For instance, if we understand that students construct their knowledge from what they already know, it is essential to constantly assess what they know and create experiences that will foster intellectual growth. That is why assessment and teaching must go hand in hand. Assessment is an integral part of the teaching and learning process, from planning to analysing its effectiveness. In fact, the main objective of assessment should be to improve teaching.

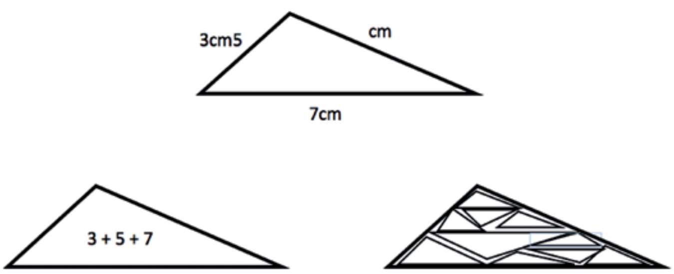
There are various ways to assess teaching, as well as what students know and what they can learn.[3,4,5] Yet, in each of them it is essential to observe what students do and listen to what they say. Moreover, we should try to understand the logic of their reasoning and explanations.

One of the challenges of assessment is being able to understand what students are thinking. This is not an easy task since often the students' logic is different from that of teachers. It is very easy to listen to and interpret from our perspective, and from there immediately try to correct what we perceive as an error, when in reality, in the students' logic, the answer is not incorrect.

Understanding the logic of learning mathematics gives us information about the stages through which our pupils will go, which, in turn, helps us to interpret their reasoning. In this sense, something we may consider a mistake—in whatever stage the child might be—could reflect sound reasoning.

For instance, we once found the following two answers to the following problem:

Find the area of this shape.



One child added the measures for all sides, and the other student made small triangles and added them. Although neither obtained the correct answer, the first made an error by confusing area with perimeter, whereas the second has an idea of what area is and how to measure it. In the first case, it is necessary to point out the confusion; in the second, we use what the student knows to show the need to standardise the units of measurement.

This example illustrates the importance of analysing students' responses and thinking about their reasoning. Another way to do this is through open problems—problems with more than one solution. For example:

The Rivera family went to a music concert. The family consists of Dad, Mum, ten-year-old Luis and eight-year-old Mariana. If they paid £50 in tickets for the entire family, how much did each ticket cost?

What possible solutions does this problem have? There are different criteria we can use to decide the price of the tickets. For example, we can say that each adult paid £15 and each child £10. We can also divide the cost evenly. This implies that we must understand decimals, since the answer is £12.50 per ticket. In fact, there are infinitely many solutions to this problem. Observing the strategies students choose allows us to see their different aptitude levels.

There may of course be students who don't know what to do when presented with this question. While observing children during problem-solving activities, we should not limit ourselves to what they can answer on their own. We should pose questions that will help them understand and learn. Knowing what students know is as important as assessing what they can learn.

The power of problem-solving

For children to learn maths with meaning, we as their teachers need to follow the logic of mathematics learning. But before we can do that, we must first understand it for ourselves. Our new book, *Math makes sense!*, explores the logic of learning of a range of mathematical concepts, from addition and subtraction, to fractions, decimals and percentages, to probability and statistics. The aim of the book is to bridge the gap between children's real-world knowledge and abstract mathematical concepts, and we present many real-life situations and models to use in your own classroom. You will also find a range of puzzles and problems, like the ones featured here, so you can introduce your pupils to the thrilling adventure made possible by mathematics.

Yes, the learning of maths must follow children's cognitive development, but this must take place in a community that promotes genuine interest in problem-solving—in making conjectures, and in sharing, discussing and arguing those conjectures with their peers. This environment must be alive and support pupils' participation in the process of using mathematics with understanding, in order to comprehend situations that are of interest and relevant to them. Only then will maths truly make sense.

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Knowledge trails

1. **Using maths to know maths –** Professor Dave Pratt seeks out mathematical methods that, when put to use, lead to natural understanding.
   library.teachingtimes.com/articles/usingmathstoknowmaths

2. **How do you measure the height of pyramids? –** The Pathway Method helps children discover answers for themselves. Peter Worley explains.
   library.teachingtimes.com/articles/height-of-pyramids