Polytopes in 4 Dimensions

A *polytope* is a polyhedron in any dimension, not necessarily dimension 3.

- A 4-dimensional polytope has vertices, edges, faces and 3-dimensional "hyperfaces" (V, E, F, H).
- A 5-dimensional polytope has vertices, edges, faces, hyperfaces, and 4-dimensional "spaces" (V, E, F, H, S).

1 Warm-up Problems

1. Last time we saw that, according to a table, there are supposed to be 2 different 3-d polyhedra with 6 faces, 12 edges, and 8 vertices. The cube is one of them. What is the other?
2. A hypercube is a 4-dimensional cube. What are V, E, F, and H for a hypercube?

2 4-Dimensional Platonic Solids

3. How would you define Platonic solids in 4-dimensions? Can you give some examples?
4. These are some vital statistics for the 4-d Platonic solids.

V	E	F	H
5	10	10	5
16	32	24	8
8	24	32	16
24	96	96	24
600	1200	720	120
120	720	1200	600

   They are called the 4-simplex, the hypercube, the 4-orthoplex, the 24-cell (or octaplex), the 120-cell, and the 600-cell.

   What patterns do you notice?

5. Build models of the 4-dimensional Platonic solids. You will actually be building their *projections*, or shadows, in 3-dimensions.
6. Find V, E, F, H for a pyramid over a 3-dimensional tetrahedron, cube, icosahedron, and dodecahedron.
7. Find V, E, F, H for a bipyramid over an octahedron.

8. What about V, E, F, H for a prism over a cube? Think about why this is the same thing as a square"times" a square.

9. What are V, E, V, H for a pentagon "times" a pentagon. Can you find formulas for V, E, F, H for the product of an m-gon and an n-gon?

And Beyond

10. What Platonic solids can you describe in 5 dimensions?

11. Calculate V, E, F, H, S for a 5-dimensional polytope, where S is the number of 4-dimensional spaces, and use these numbers to find the Euler characteristic $V - E + F - H + S$.

12. Which polytopes generalize easily to every dimension? What is Euler's formula in dimension n?

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