Polytopes in 4 Dimensions

A polytope is a polyhedron in any dimension, not necessarily dimension 3.

- A 4-dimensional polytope has vertices, edges, faces and 3-dimensional “hyperfaces” $(V, E, F, H)$.
- A 5-dimensional polytope has vertices, edges, faces, hyperfaces, and 4-dimensional “spaces” $(V, E, F, H, S)$.

1 Warm-up Problems

1. Last time we saw that, according to a table, there are supposed to be 2 different 3-d polyhedra with 6 faces, 12 edges, and 8 vertices. The cube is one of them. What is the other?
2. A hypercube is a 4-dimensional cube. What are $V$, $E$, $F$, and $H$ for a hypercube?

2 4-Dimensional Platonic Solids

3. How would you define Platonic solids in 4-dimensions? Can you give some examples?
4. These are some vital statistics for the 4-d Platonic solids.

<table>
<thead>
<tr>
<th>$V$</th>
<th>$E$</th>
<th>$F$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>32</td>
<td>24</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>32</td>
<td>16</td>
</tr>
<tr>
<td>24</td>
<td>96</td>
<td>96</td>
<td>24</td>
</tr>
<tr>
<td>600</td>
<td>1200</td>
<td>720</td>
<td>120</td>
</tr>
<tr>
<td>120</td>
<td>720</td>
<td>1200</td>
<td>600</td>
</tr>
</tbody>
</table>

They are called the 4-simplex, the hypercube, the 4-orthoplex, the 24-cell (or octa-plex), the 120-cell, and the 600-cell.

What patterns do you notice?
5. Build models of the 4-dimensional Platonic solids. You will actually be building their projections, or shadows, in 3-dimensions.
6. Find $V, E, F, H$ for a pyramid over a 3-dimensional tetrahedron, cube, icosahedron, and dodecahedron.
7. Find $V, E, F, H$ for a bipyramid over an octahedron.
8. What about $V, E, F, H$ for a prism over a cube? Think about why this is the same thing as a square “times” a square.

9. What are $V, E, V, H$ for a pentagon “times” a pentagon. Can you find formulas for $V, E, F, H$ for the product of an $m$-gon and an $n$-gon?

**And Beyond**

10. What Platonic solids can you describe in 5 dimensions?

11. Calculate $V, E, F, H, S$ for a 5-dimensional polytope, where $S$ is the number of 4-dimensional spaces, and use these numbers to find the Euler characteristic $V - E + F - H + S$.

12. Which polytopes generalize easily to every dimension? What is Euler’s formula in dimension $n$?

Thanks to Matt Kahle for most of these problems.