# Math Circle Worksheet
# Fun with Matrices a.k.a. Basic Linear Algebra

Wesley Hamilton

4/08/2017

## 1 Introduction

Linear algebra pops up everywhere in math, sometimes in an obvious way, sometimes in a sneaky way. One example is calculus! Linear algebra is the study of lines, planes, and higher dimensional analogues, whereas differential calculus is (more-or-less) the study of tangent lines to curves. Many topics in combinatorics can be rephrased using matrices, groups (which were the topic of an earlier math circle) can be studied using linear algebra, and the list continues. On the applied side, Google's page-rank algorithm is actually a problem involving matrices and, as such, is an application of linear algebra.

This worksheet will give you a taste of what linear algebra is, and how we can get some interesting results (some may be familiar) in ways you might not expect. For example, we can find a closed form expression for Fibonacci numbers using a certain matrix and its powers. We'll also play around with some geometry, look at transformations on the plane, specialize to rotations, and finally look at the connection between rotations and complex numbers.

## 2 Basics on Matrices and Matrix Operations

In this section we'll introduce the concept of a matrix, as well as how to work with them. This section is just a collection of tools and won't seem very motivated. Don't worry though, we'll see plenty of cool ways that matrices are utilized in the later sections.

### 2.1 What is a matrix?

The most basic interpretation of a matrix is a rectangular array of numbers. For example, these are all matrices:

$$\begin{pmatrix} 3 & 7 \\ 0.5 & 2 \end{pmatrix}, \begin{pmatrix} 2 & \pi & 2\pi \\ \sqrt{2} & \frac{1}{\pi} & 300 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 \end{pmatrix}, \begin{pmatrix} 1.12345 \end{pmatrix}.$$

There isn't any extra structure we put on these arrays, they are just organized collections of numbers. In what follows, we'll usually refer to the numbers of the array as *elements* of the matrix. The first matrix above is a $2 \times 2$ matrix, since it has two rows and two columns. The second is a $2 \times 3$ matrix, since it has two rows and three columns.

**Exercise 2.1** What kinds of matrices are the last two above? In other words, they are $m \times n$ matrices where $m$ and $n$ are integers; what are the integers?

In many situations, we think of $1 \times 1$ matrices as a real number, since they are more-or-less the same kind of object.

## 2.2 How do we add matrices?

Now we'll put a little more structure on these objects, starting with addition. For two matrices of the same type (say, two $2 \times 2$ matrices, or two $3 \times 4$ matrices, etc.), we define addition element-wise. These examples should make this clear:

$$\begin{pmatrix} 2 & 5 & 8 \\ 1 & \pi & 0 \end{pmatrix} + \begin{pmatrix} 1 & 5 & -2 \\ 0 & 3\pi & 1 \end{pmatrix} = \begin{pmatrix} 2+1 & 5+5 & 8+(-2) \\ 1+0 & \pi+3\pi & 0+1 \end{pmatrix} = \begin{pmatrix} 3 & 10 & 6 \\ 1 & 4\pi & 1 \end{pmatrix},$$
$$\begin{pmatrix} 3 & 4 \\ 2 & -7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 18 \end{pmatrix} = \begin{pmatrix} 3+1 & 4+2 \\ 2+0 & -7+18 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 11 \end{pmatrix}.$$

This is why we need the matrices to be of the same type, since otherwise one of the elements might not have a corresponding element to be added to.

**Exercise 2.2** What matrix do you get from adding $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

**Exercise 2.3** What is the additive inverse of the matrix $\begin{pmatrix} 7 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$? In other words, find the matrix $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ such that

$$\begin{pmatrix} 7 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Exercise 2.4** Using the last exercise (or otherwise), find the additive inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d$ can be any real number.

We call the matrix filled with 0s the zero matrix (who would've thought), and write it as just 0. For example, the $2 \times 2$ zero matrix is $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Another important matrix (especially in the upcoming sections) is the *identity matrix*, which we'll write as $I$. This is the $m \times m$ matrix with 1s along the diagonal and 0s elsewhere. In the $2 \times 2$ case, it looks like $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

## 2.3 How do we multiply matrices?

Now we'll introduce an even fancier structure to our collection of matrices, multiplication. If $A$ is an $m \times n$ matrix and $B$ is an $n \times o$ matrix, then we define their *product* in a slightly strange way, which I'll show with some examples. It's important to note that the number of columns of $A$ has to be the same as the number of rows of $B$!

Explicitly for $2 \times 2$ matrices (which we'll mainly be using):

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 8 & 0 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 8 + 2 \cdot (-1) & 1 \cdot 0 + 2 \cdot 4 \\ 3 \cdot 8 + 2 \cdot (-1) & 3 \cdot 0 + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 22 & 8 \end{pmatrix},$$

$$\text{and so in general } \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 + a_2 \cdot b_3 & a_1 \cdot b_2 + a_2 \cdot b_4 \\ a_3 \cdot b_1 + a_4 \cdot b_3 & a_3 \cdot b_2 + a_4 \cdot b_4 \end{pmatrix}.$$

For other types of matrices multiplication takes the same kind of form, where we match elements in the rows of the first matrix with elements in the column of the second matrix, then add them up. Another type of matrix multiplication that we'll make extensive use of is a $2 \times 2$ matrix by a $2 \times 1$ matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-2) + 2 \cdot 4 \\ 3 \cdot (-2) + 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

$$\text{and so in general } \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 + a_2 \cdot b_2 \\ a_3 \cdot b_1 + a_4 \cdot b_2 \end{pmatrix}.$$

Since we don't need to multiply other types of matrices, I won't go into the general case, but here's an example of a $2 \times 2$ matrix multiplied with a $2 \times 3$ matrix (notice that the dimensions match up in the way they should!):

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 8 & 0 & 1 \\ -1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 8 + 2 \cdot (-1) & 1 \cdot 0 + 2 \cdot 4 & 1 \cdot 1 + 2 \cdot 3 \\ 3 \cdot 8 + 2 \cdot (-1) & 3 \cdot 0 + 2 \cdot 4 & 3 \cdot 1 + 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 7 \\ 22 & 8 & 9 \end{pmatrix}.$$

**Exercise 2.5** Compute $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$. What is $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix}$? Notice any connections between these two matrix products?

**Exercise 2.6** Compute $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \cdot I = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. What do you notice?

The last exercise shows why we call $I$ the identity matrix; it's the multiplicative identity for matrices (just like how 1 is the multiplicative identity for real numbers)!

**Exercise 2.7** Compute $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. What do you notice?

**Exercise 2.8** What about $\begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$? What is the difference between matrix multiplication and ordinary real number multiplication?

**Exercise 2.9** Given two $2 \times 2$ matrices $A, B$, what conditions do you need for $A \cdot B = B \cdot A$? If you're not sure, try a simpler case: what if $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$? Do these commute?

In one of the last exercises you should have gotten the 0 matrix. Don't be alarmed! There are other places in math where multiplying two non-zero things gives zero, and matrix multiplication is one of those places. So long as we don't divide by zero, we won't run into any trouble.

Finally, we have another kind of multiplication for matrices. This time though we multiply the matrix by a real number, not another matrix, and the multiplication is done element-wise (just like addition). For example,

$$3 \cdot \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2 & 3 \cdot (-2) \\ 3 \cdot (-2) & 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}.$$

## 2.4 The inverse of a $2 \times 2$ matrix

Now that we know how to add, subtract, and multiply matrices together, let's look into division.

When we divide two real numbers, say $x/y$, what we're really doing is looking for a number $z$ such that $x = yz$. For example, $4/2$ (four divided by two) is equal to 2, because $4 = 2 \cdot 2$. A good starting point is to find the inverse of numbers, like the multiplicative inverse of 2, and define division as the multiplication by the inverse. This is the approach we'll take.

To this end, we need to know what the inverse of a $2 \times 2$ matrix is: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$, then the **inverse** of $A$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Exercise 2.10** Check that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc \neq 0$, then we do indeed have

$$A \cdot A^{-1} = I.$$

Thus, if we want to divide the matrix $A$ by the matrix $B$, we multiply $A \cdot B^{-1}$. The only real number we can't divide by is 0, and this is encapsulated by the requirement that $\det(A) = ad - bc \neq 0$. Whenever you want to check whether something is invertible, compute the **determinant**[1] $\det(A)$ and see if that's 0.

**Exercise 2.11** Is $I$ invertible? Why or why not? What about 0?

[1] The determinant is a funny function that may seem completely out of the blue and unmotivated. At this stage you're not wrong, but there are some deeper, more natural reasons as to why the determinant is defined the way it is (and it's an incredibly important function when working with matrices).

**Exercise 2.12** Is $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$ invertible? Why or why not?

Look back at exercise 2.7, where the last matrix first appeared. It isn't a coincidence that the product there was 0, and that the matrix isn't invertible. In fact, if we only look at matrices that satisfy $\det(A) = ad - bc \neq 0$, i.e. the set

$$GL_2 = \{2 \times 2 \text{ matrices } A \text{ that satisfy } \det(A) \neq 0\},$$

then with matrix multiplication we get a group! Neat! In the following sections, it turns out to be this group structure that gives us all of the cool properties that we'll explore.

**Exercise 2.13** * Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$. Let $A'$ be the matrix we get by switching the columns of $A$. Show that $\det(A) = -\det(A')$.

**Exercise 2.14** * Suppose that one of the columns of $A$ is a multiple of the other, say $\begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = c \begin{pmatrix} a_2 \\ a_4 \end{pmatrix}$ for some constant $c$. Show that in this case $\det(A) = 0$.

**Exercise 2.15** * Show that, if $A, B$ are $2 \times 2$ matrices, then $\det(A \cdot B) = \det(A) \cdot \det(B)$. Is $\det(A \cdot B) = \det(B \cdot A)$?

# 3 Fibonacci Numbers and Matrices

The Fibonacci numbers $F_n$ are defined by the recursive relation

$$F_{n+2} = F_{n+1} + F_n, \quad F_1 = 1, F_2 = 1.$$

In this section we'll find a nice expression for $F_n$ using matrices. Curiously enough, the golden ration $\frac{1+\sqrt{5}}{2}$ also plays a role.

In what follows, we'll make extensive use of the matrix $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

**Exercise 3.1** What is $F^2$? What about $F^3$? What about $F^n$? If you're not sure what the pattern is, compute $F^4, F^5$, etc. as well.

Next, we're going to introduce a "change of basis" matrix $A = \begin{pmatrix} \phi_1 & 1 \\ \phi_2 & 1 \end{pmatrix}$, where $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$. Here, $\phi_1$ and $\phi_2$ are the roots of the polynomial $x^2 - x - 1$, and $\phi_1$ is often called the *golden ratio*. The next exercise shows why we might want to introduce and use this matrix $A$.

**Exercise 3.2** What is $A \cdot F \cdot A^{-1}$? For this you need to verify that $A$ is invertible, compute the inverse, then compute the product. Hint: since $\phi_i$ is a root of $x^2 - x - 1$, we have $\phi_i^2 - \phi_i - 1 = 0$ for $i = 1, 2$.

We'll call the matrix from the last exercise $D$, so that $D = A \cdot F \cdot A^{-1}$. The matrix $D$ you computed should be diagonal (which is why we named it $D$).

**Exercise 3.3** Now compute $F^n$, as well as $D^n$.

**Exercise 3.4** Show that $(A \cdot F \cdot A^{-1})^n = A \cdot F^n \cdot A^{-1}$. Use this to show that $A \cdot F^n \cdot A^{-1} = D^n$.

**Exercise 3.5** Using the last exercise, what is a closed form expression for the $n$-th Fibonacci number? By closed form, I mean a non-recursive representation for the numbers $F_n$. Hint: $AF^nA^{-1} = D^n$ is the same equality as $F^n = A^{-1}D^nA$ (why is this true?).

**Exercise 3.6** What is the limit of the ratio of successive Fibonacci numbers $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$?

Much of the math underpinning this problem utilizes objects called **eigenvectors** and **eigenvalues**. To be precise, the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ are $\phi_1$ and $\phi_2$, because there exist vectors $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \phi_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \phi_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

**Exercise 3.7** What are $v_1, v_2, u_1, u_2$? How do they relate to $A$?