Math Circle Worksheet
Understanding Rubik's Cube

Wesley Hamilton

2/25/2017

1 The 15-Puzzle

Consider the 15-puzzle, where you have a 4×4 grid of the numbers $1, 2, ..., 15$, and the bottom right corner is empty. The goal of the puzzle is to, by sliding tiles into the empty square, transform the original configuration into the neutral configuration. A natural question (we'll call this the **important question**) is if any starting position can be transformed into the neutral configuration, and the symmetric group gives us a nice, easy answer (no, not every configuration can be).

To start, we'll play around with a few easy examples involving a simpler board: a 2×2 grid with the bottom right corner removed.

1. Suppose the 1-tile is in the 2-tile place, the 2-tile is in the 3-tile place, and the 3-tile is in the 1-tile place. Note that this corresponds to a permutation (123). Is this puzzle solvable?

2. What about the board corresponding to the permutation (132)? Is this board solvable?

3. Finally, what about the board corresponding to the transposition (12)? Is this board solvable? Why or why not?

As you solve the, say, 15-puzzle, notice that the empty square travels around the board (up and away from its starting position). One important step in our efforts to answer the **important question** is to label the empty square 16.

4. How would you express, using cycle notation, the trip the empty cell takes around our 15-puzzle grid? In other words, what permutation represents sliding a tile into the empty slot?

Instead of starting with an attainable board and trying to slide tiles around to get it to the neutral board, we could also start with the neutral board, slide the tiles around, and arrive at an attainable board.

5. If you want to make an attainable board from the neutral board, you can take successive transpositions of the empty tile with an adjacent square. In other words, you'll start constructing a permutation that looks like $\cdots (16\,11)(16\,15)$. Why?

6. If after the empty (16-)tile slides around, where does it need to end up for the board to be an attainable board? What does the corresponding permutation look like?

7. What is the parity of this permutation?

This suggests that if a board is solvable (just do the operations to get from the neutral board in reverse), then the corresponding permutation must have parity 1. What about the other direction though? If the permutation has parity 1, must the board be solvable?

The answer turns out to be yes! The proof is a little more involved, but one way to get a feel for it is to just try and solve the board. Depending on whether or not the starting permutation had parity 1, you should end up with the neutral board, or the board corresponding to the transposition (1314) (or any other transposition).

2 General Group Theory

Back to some general group theory before we analyze Rubik's cube. We know that the inverse of a group element annihilates the group element, i.e. if a is an element of $(G, \cdot)$, then $a \cdot a^{-1} = e$.

8. If a and b are in G, what is $(a \cdot b)^{-1}$? Note that it is not $a^{-1} \cdot b^{-1}$.

9. What must happen for $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$? Why?

Another key concept in understanding Rubik's cube is that of a group **generated** by some elements. The idea is, if you're given a set of elements $S = \{a_1, ..., a_n\}$ then the group generated by S, which we usually write as $\langle S \rangle$, is the collection of all combinations of the a_i, together with their inverses.

10. Consider the group $(\mathbb{Z}, +)$, and take the group element 4. What is $\langle 4 \rangle$?

11. What about $\langle 6, 4 \rangle$?

12. Finally, what is $\langle 9, 4 \rangle$?

13. * In general if you have integers $n_1, ..., n_m$, what is $\langle n_1, ..., n_m \rangle$?

14. Now take the group $(\mathbb{Q}^*, \times)$. What is $\langle \frac{1}{2} \rangle$? What about $\langle 2 \rangle$?

The next few exercises show a slightly more interesting example of a group generated by some group elements. Recall that S_n is the group of permutations of the set $\{1, ..., n\}$. We'll also need the fact that and permutation σ in S_n can be written as a product of transpositions.

15. Is $\langle (12), (13), ..., (1n), (23), ..., (n-1\,n) \rangle = S_n$? In other words, is S_n generated by all transpositions?

16. * Is there a smaller generating set, still consisting of permutations, for S_n? Why or why not?

3 Rubik's Cube

Rubik's cube is a puzzle invented about 30 years ago by Enriko Rubik. Coincidentally, Rubik was influenced by the 15-puzzle, which itself was invented in the late 1800s. The cube has 6 faces of 9 squares each, and each square has one of 6 colours. The goal of the puzzle is to, using rotations of the faces, "solve" the puzzle by getting all squares of the same colour to the same face.

To analyze how Rubik's cube works we first need to come up with some terminology, specifically we need to name each of the "cubies" (the small faces) and each of the rotations.

Start by holding the cube so that its face is directly facing you. We'll call the entire face of the cube F, for *front*. The face *behind* this face will be called B. Similarly, the faces to the *left* and *right* of F will be called L and R, respectively. Finally, we'll call the *upper* face U and the lower face D (for *down*).

As for the cubies, we'll name them by their relation to the center of a face. First we'll name the center cubies by the face they're on, of which there are 6: $[F], [B], [L], [R], [U], [D]$.

The edge cubies (pieces of the cube that have two colours/only have an edge) will be labelled by the faces they touch. For example, the cubie above the center cubie $[F]$ will be called $[FU]$, while the cubie on $[U]$ closest to F will be called $[UF]$. Note that $[UF]$ is not the same cubie as $[FU]$, and this distinction is important.

Corner cubies will be denoted by three letters: the face the cubie is on, followed by the two adjacent faces. For example, the cubie on the face F, adjacent to the faces U and L, will be written $[ULF]$.

17. It turns out that $[ULF]$ is the same cubie as $[UFL]$. Why is this the case?

Now we need to name the rotations. Notice that any move we make on Rubik's cube is generated by a rotation of one of the faces. Thus, we'll just name these basic rotations: a rotation of the face F clockwise by 90 degrees will be called F, a rotation of the face U by 90 degrees will be called U, etc.

We can think of (and this is a very fruitful mindset) these face rotations as permutations of cubies. Here, though, we're reading off the permutations from left to right (opposite what we did with the 15-puzzle). So FR means perform F first, then perform R. In the 15-puzzle, we would have performed R first followed by F.

Thus, all the allowed moves are generated by one of these rotations (plus its inverse), so $\mathfrak{R}$, the set of Rubik's cube permutations, is precisely

$$\mathfrak{R} = \langle F, B, U, D, L, R \rangle.$$

18. What permutation of cubies does the rotation F correspond to? What about R? It's highly recommended you use cycle notation here.

19. What permutation of cubies is the action $FFRR$?

20. What about $(FFRR)^3$?

21. Interpret your answer on a cube. What happens?

Permutations like $(FFRR)^3$ are called **macros**, since they're combinations of simple moves that do something useful. Algorithms for solving Rubik's cube usually rely on sequences of macros.

If we want the inverse of a rotation, we'll write the corresponding lower case letter. In other words, f will be the rotation of the front face F counterclockwise by 90 degrees. Symbolically, $Ff = fF = e$, the identity.

22. What is the cycle decomposition of $FUllR$? Remember that $l = L^{-1}$.

23. What is $(FUllR)^9$?

24. What is $(FUllR)^{18}$?

Neat! Thinking of these macros as permutations leads to a few other interesting conclusions.

25. Is it possible to have a transposition (any transposition of cubies) as a macro? Why or why not? Hint: what is the parity of a transposition?

26. What about any 3-cycle? Why or why not?